Berkeley Math Circle: Monthly Contest 6 Solutions

1. Ten fair coins are flipped. Given that there are at least nine heads, what is the probability that all the coins show heads?

   Solution. The answer is $\frac{1}{11}$. Among the $2^{10} = 1024$ sequences of heads and tails, note that

   - There is only 1 which is all heads, and
   - There are 10 sequences which have exactly one tails.

   So there are 11 possible sequences with at least nine heads, only one of which is all heads. Hence the answer is $\frac{1}{11}$. $\square$

2. Is there a positive integer n for which $n(n+1)$ is a perfect square?

   Solution. The answer is no. In fact, we note the number above is sandwiched between two consecutive perfect squares:

   $$n^2 < n(n+1) < n^2 + 2n + 1 = (n+1)^2.$$

   So it cannot itself be a perfect square. $\square$

3. Prove that for any positive integer n, we have

   $$\prod_{k=1}^{n} \operatorname{lcm}\left(1, 2, \dots, \left\lfloor \frac{n}{k} \right\rfloor\right) = n!.$$

   Solution. We show that the exponents of p coincide for any prime p. Actually, we'll prove the stronger claim that for any prime power q, the number of terms on each side divisible by q is exactly the same. By the fundamental theorem of arithmetic, that will imply the desired equality.

   Clearly, the number of terms on the right-hand side which are divisible by q is $\lfloor n/q \rfloor$.

   As for the left-hand side, the number of lcm's on the right which are divisible by q is given by the number of k for which $\left\lfloor \frac{n}{k} \right\rfloor \geq q$, which is exactly $k = 1, \dots, \lfloor n/q \rfloor$. $\square$

4. Let ABC be a triangle and let P be a point inside it satisfying $\angle ABP = \angle PCA$. Let Q be the reflection of P across the midpoint of $\overline{BC}$. Prove that $\angle BAP = \angle CAQ$.

   Solution. Construct parallelogram $APBR$, so that segments AR, BP, QC are congruent and parallel. Then $ARQC$ is a parallelogram as well. We contend now that $ARBQ$ is cyclic. Indeed,

   $$\begin{aligned} \angle ARQ &= \angle ACQ \\ &= \angle ACP + \angle PCQ \\ &= \angle QBP + \angle PBA \\ &= \angle QBA. \end{aligned}$$

Finally

$$\angle BAP = \angle ABR = \angle AQR = \angle QAC.$$

$\square$

5. Find the smallest prime $p > 100$ for which there exists an integer $a > 1$ such that p divides $\frac{a^{89}-1}{a-1}$.

   Solution. The answer is $p = 179$. To see this works, take $a = 4$; by Fermat's little theorem, $4^{89} - 1 = 2^{178} - 1$ is divisible by 179.

   Now suppose $a^{89} \equiv 1 \pmod p$. We consider two cases:

   - If $a \equiv 1 \pmod p$, then

     $$0 \equiv 1 + a + \cdots + a^{88} \equiv 89 \pmod p$$

     which forces $p = 89$.
   - Otherwise, since 89 is prime, it follows a has order 89 modulo p. So $89 \mid p - 1$. The smallest prime which obeys this is $p = 179$. $\square$

6. Convex quadrilateral $ABCD$ with $BC = CD$ is inscribed in circle Ω; the diagonals of $ABCD$ meet at X. Suppose $AD < AB$, the circumcircle of triangle BCX intersects segment AB at a point $Y \neq B$, and ray $\overrightarrow{CY}$ meets Ω again at a point $Z \neq C$. Prove that ray $\overrightarrow{DY}$ bisects angle ZDB.

   Solution. This is mostly just angle chasing. In this case Y and Z lie between A and B, on the respective segment/arc. We'll prove Y is the incenter of $\triangle ZDB$; it will follow that ray $\overrightarrow{DY}$ indeed internally bisects $\angle ZDB$. It suffices to prove the following two facts:

   - BY is the internal angle bisector of $\angle DBZ$. This is true in general; it doesn't require $CB = CD$. It's part of the spiral similarity configuration centered at $B : YX \to ZA$ and $B : ZY \to AX$, due to $YZ \cap AX = C$ and $B = (CYX) \cap (CZA)$. More explicitly, this follows from the angle chase

     $$\angle DBA = \angle XBY = \angle XCY = \angle ACZ = \angle ABZ.$$

   - ZY is the internal angle bisector of $\angle BZD$, since $CB = CD$. Indeed (more explicitly), arcs BC and CD are equal, so $\angle BZC = \angle CZD$, i.e. YZ bisects $\angle BZD$. $\square$

7. Prove that there are infinitely many positive integers n for which $n^2 + 1$ has no repeated prime factors (that is, $n^2 + 1$ is *squarefree*).

   Solution. By Fermat's Christmas theorem, the only primes which may divide $n^2 + 1$ other than 2 are those which are $1 \pmod 4$, and moreover $2^2 \nmid n^2 + 1$ for any n.

   Consider primes $p \equiv 1 \pmod 4$. Observe that for any x, we have

   $$\#\{n \le x \mid n^2 + 1 \equiv 0 \pmod{p^2}\} \le \frac{2}{p^2}x + 2$$

since there are at most two solutions mod p^2 to $n^2 \equiv -1 \pmod{p^2}$. Summing over all primes $p \equiv 1 \pmod 4$ now implies the result, since

$$\sum_{p \equiv 1 \pmod 4} \frac{2}{p^2} < 2\left(\frac{1}{4^2} + \frac{1}{8^2} + \dots\right) = \frac{\pi^2}{6} \cdot \frac{1}{8} < 1. \qquad \square$$