Berkeley Math Circle: Monthly Contest 4 Solutions

1. When the number
$$N = 1^1 \times 2^2 \times 3^3 \times \cdots \times 9^9$$
is written as a decimal number, how many zeros does it end in?

   Solution. The number N ends with five zeros. Indeed, 10^5 divides N since $4^4 \times 5^5 = 800{,}000$ divides N. But 10^6 does not divide N, since the only terms in the product which are divisible by 5 are 5^5, and hence 5^6 does not divide N. So the answer is 5. □

2. A square and an equilateral triangle have the property that the area of each is the perimeter of the other. What is the area of the square?

   Solution. Assuming the square has side length x, it has area x^2, so the equilateral triangle has side length $x^2/3$. The area of the equilateral triangle is then given in two ways by
$$4x = \sqrt{3}/4 \cdot \left(\frac{x^2}{3}\right)^2.$$
Solving gives $x^3 = \frac{144}{\sqrt{3}} = 48\sqrt{3}$. Then $x^6 = 48^2 \cdot 3$ and $x^2 = \sqrt[3]{48^2 \cdot 3} = \sqrt[3]{2^8 3^3} = 12\sqrt[3]{4}$. □

3. Find all the ways which one can assign an integer to each vertex of a 100-gon subject to the following condition: among any three consecutive numbers written down, one of the numbers is the sum of the other two.

   Solution. The answer is that all the numbers must be zero. (Clearly, this works.)

   We now prove this is the only solution. Call the numbers x_1, x_2, …, x_{100}. Then the sum $x_1 + x_2 + x_3$ must be even, since it is either $2x_1$, $2x_2$, or $2x_3$. Similarly, $x_2 + x_3 + x_4$ must be even.

   In this way, x_1 and x_4 have the same parity. By the same reasoning, x_4 and x_7 have the same parity, and so on — the numbers x_k and x_{k+3} have the same parity. Since 3 doesn't divide 100, that means all the numbers have the same parity. Clearly then all the numbers are even (rather than all odd).

   We may now employ infinite descent: if $(x_1/2, \dots, x_{100}/2)$ is a working assignment, then so is $(x_1/2, x_2/2, \dots, x_{100}/2)$, and then so is $(x_1/4, x_2/4, \dots, x_{100}/4)$. Such a process cannot go on indefinitely unless $x_k = 0$ for all k, completing the proof. □

4. Give an example of a *strictly increasing* function $f : \mathbb{R} \to [0, 1]$ with the property that
$$f(x + y) \le f(x) + f(y)$$
for any real numbers x and y.

Solution. We claim that

$$f(x) = \frac{1}{1+e^{-x}}$$

works fine. This function is strictly increasing by definition, so all that's left to do is check the inequality

$$\frac{1}{1+e^{-(x+y)}} \le \frac{1}{1+e^{-x}} + \frac{1}{1+e^{-y}}.$$

Letting $a = e^{-x}$ and $b = e^{-y}$ it's the same as to check

$$\frac{1}{1+a} + \frac{1}{1+b} \ge \frac{1}{1+ab}$$

where $a, b > 0$. But

$$\begin{aligned} \frac{1}{1+a} + \frac{1}{1+b} - \frac{1}{1+ab} &= \frac{(2+a+b)(1+ab) - (1+a)(1+b)}{(1+a)(1+b)(1+ab)} \\ &= \frac{1+ab+ab(a+b)}{(1+a)(1+b)(1+ab)} \\ &> 0. \end{aligned}$$

□

5. Louis moves around on the lattice points according to the following rules: From point (x, y) he may move to any of the points (y, x), $(3x, -4y)$, $(-2x, 5y)$, $(x+1, y+6)$ and $(x-7, y)$. Show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

   Solution. Call a point (x, y) stable if $x + y$ is not divisible by 7. The key is to observe that starting from a stable point, one may only reach other stable points. For example, $3x - 4y \equiv 3(x + y) \pmod 7$, hence if (x, y) is stable then $(3x, -4y)$ is as well.

   Consequently, starting from the stable point $(1, 0)$ it's impossible to reach the unstable point $(0, 0)$. □

6. A sequence a_1, a_2, . . . of positive integers satisfies $a_1 = 1$ and

   $$a_{n+1} = 2^{a_n} + a_n$$

   for $n \ge 1$. Prove that a_1, a_2, . . . , a_{243} leave distinct remainders when divided by 243.

   Solution. I'll prove by induction on $k \ge 1$ that any 3^k consecutive values of a_n produce distinct residues modulo 3^k. The base case $k = 1$ is easily checked (a_n is always odd, hence a_n cycles 1, 0, 2 mod 3).

   For the inductive step, assume it's true up to k. Since $2^* \pmod{3^{k+1}}$ cycles every $2 \cdot 3^k$, and a_k is always odd, it follows that

   $$\begin{aligned} a_{n+3^k} - a_n &= 2^{a_n} + 2^{a_{n+1}} + \cdots + 2^{a_{n+3^k-1}} \pmod{3^{k+1}} \\ &\equiv 2^1 + 2^3 + \cdots + 2^{2\cdot 3^k - 1} \pmod{3^{k+1}} \\ &= 2 \cdot \frac{4^{3^k} - 1}{4 - 1}. \end{aligned}$$

Hence

$$a_{n+3^k} - a_n \equiv C \pmod{3^{k+1}} \qquad \text{where} \qquad C = 2 \cdot \frac{4^{3^k} - 1}{4 - 1}$$

noting that C does not depend on n. Exponent lifting gives $\nu_3(C) = k$ (meaning 3^k divides C but not 3^{k+1}) hence a_n, a_{n+3^k}, $a_{n+2\cdot 3^k}$ differ mod 3^{k+1} for all n, and the inductive hypothesis now solves the problem. □

7. Let ABC be a triangle with incenter I and circumcenter O for which $BC < AB < AC$. Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Prove that $\overline{DE} \perp \overline{IO}$.

Solution. It is enough to show that $DI^2 - DO^2 = EI^2 - EO^2$. But if we let R denote the circumradius, then by power of a point we have $R^2 - DO^2 = AD \cdot DB$, and $R^2 - EO^2 = AE \cdot EC$. Thus it suffices to prove

$$DI^2 + AD \cdot DB = EI^2 + AE \cdot EC \iff DI^2 - EI^2 = AE \cdot EC - AD \cdot DB.$$

In the usual notation $a = BC$, $b = CA$, $c = AB$, the right-hand side is

$$AE \cdot EC - AD \cdot DB = (b - a)a - (c - a)a = a(b - c).$$

Now let the foot from I to BC be K; it's well-known that $BK = \frac{1}{2}(a - b + c)$ and $CK = \frac{1}{2}(a + b - c)$. So

$$\begin{aligned} DI^2 - EI^2 &= CI^2 - BI^2 \\ &= (IK^2 + CK^2) - (IK^2 + BK^2) \\ &= (CK - BK)(CK + BK) \\ &= a(c - b). \end{aligned}$$

This gives the desired equality, so we're done. □