Berkeley Math Circle: Monthly Contest 2 Solutions

1. You are out walking and see a group of rhinoceroses (which each have two horns and four legs) and triceratopses (which each have three horns and four legs). If you count 31 horns and 48 legs, how many triceratopses are there?

   Solution. Since each animal has 4 legs, there must be $48/4 = 12$ animals. Each triceratops has 3 horns and each rhinoceros has 2, so if there are t triceratopses and r rhinoceroses we get $t+r = 12$ and $3t+2r = 31$. Subtracting twice the first equation from the second gives $t = 7$. □

2. Let n be a positive integer. There are n islands, and you want to build some number of bridges so that there is a path between any two islands. (Each bridge joins a pair of islands, and can be traveled in both directions.) At least how many bridges must you build?

   Solution. You need to build at least $n - 1$ bridges. Imagine you start at one island and are only building bridges that start from islands you can already reach. Then, each new bridge connects you to only one new island, so to be connected to all of them, you need $n - 1$ bridges. Conversely, $n - 1$ bridges is enough, since you can go from island 1 to island 2, then from island 2 to island 3 and so on in one path until you have reached all the islands. □

3. Let n be a nonnegative integer. Prove that the numbers $n+2$ and n^2+n+1 cannot both be perfect cubes.

   Solution. If both numbers are perfect cubes then so is their product. But

   $$(n + 2)(n^2 + n + 1) = n^3 + 3n^2 + 3n + 2 = (n + 1)^3 + 1,$$

   which cannot be a perfect cube, contradiction. □

4. The endpoints of a chord ST with constant length are moving along a semicircle with diameter AB. Let M be the midpoint of ST and P the foot of the perpendicular from S to AB. Prove that the angle SPM is independent of the location of ST.

   Solution. Draw the other half of the circle, and extend SP until it hits the circle again at S'. Note that S' is the reflection of S across AB. Then by SAS, $\triangle PSM \sim \triangle S'ST$, so $\angle SPM = \angle SS'T$. But $\angle SS'T$ is constant, since it is inscribed in arc ST of the circle, so $\angle SPM$ is constant as well. □

5. Let n be a positive integer. Is it possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the arithmetic mean of any two of these numbers is not equal to some number between them?

Solution. This is possible for every n. Note that if it is possible for some number greater than n, then it is also possible for n. Thus, it suffices to prove the statement for powers of 2. We will proceed by induction. First, the statement is true for 2, since 1, 2 works. Now suppose it is possible for 2^k, and the sequence $a_1, a_2, \ldots, a_{2^k}$ works. Then for 2^{k+1}, we can use the sequence

$$2a_1, 2a_2, \ldots, 2a_{2^k}, 2a_1 - 1, 2a_2 - 1, \ldots, 2a_{2^k} - 1,$$

with all the even numbers in the first half and all the odd numbers in the second half. If two numbers are in the same half, then by the inductive hypothesis their arithmetic mean cannot be between them. If two numbers are in different halves, then they have different parity so their arithmetic mean is not an integer and therefore not in the sequence. Thus, this sequence works, so by induction the statement holds for all 2^k and thus also for all n. □

6. Let a, b, x, y, z be positive real numbers. Prove the inequality

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq \frac{3}{a+b}.$$

Solution. Applying Cauchy-Schwarz inequality to the triples

$$\sqrt{\frac{x}{ay+bz}}, \sqrt{\frac{y}{az+bx}}, \sqrt{\frac{z}{ax+by}} \quad \text{and} \quad \sqrt{x(ay+bz)}, \sqrt{y(az+bx)}, \sqrt{z(ax+by)},$$

we get that

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq \frac{(x+y+x)^2}{(a+b)(xy+yz+zx)}.$$

But we note that

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 0 \implies (x+y+z)^2 \geq 3(xy+yz+zx),$$

and the desired inequality follows. □

7. Let $a_0 = a_1 = 1$ and $a_{n+1} = 7a_n - a_{n-1} - 2$ for all positive integers n. Prove that a_n is a perfect square for all n.

Solution. We claim that $a_n = F_{2n-1}^2$, where F_n is the nth Fibonacci number. For the base case, we compute the first four values:

$$a_1 = 1^2 = F_1^2, \quad a_2 = 7 \cdot 1 - 1 - 2 = 2^2 = F_3^2,$$

$$a_3 = 7 \cdot 4 - 1 - 2 = 5^2 = F_5^2, \quad a_4 = 7 \cdot 25 - 4 - 2 = 13^2 = F_7^2.$$

Now assume the statement holds for all $1 \leq k \leq n$ for $n \geq 4$. For the inductive step, we subtract $a_n = 7a_{n-1} - a_{n-2} - 2$ from $a_{n+1} = 7a_n - a_{n-1} - 2$ to get

$$a_{n+1} = 8a_n - 8a_{n-1} + a_{n-2} = 8F_{2n-1}^2 - 8F_{2n-3}^2 + F_{2n-5}^2.$$

But we find that for any $m \geq 2$,

$$F_{m-2} = F_m - F_{m-1} = F_m - (F_{m+1} - F_m)$$

$$= 2F_m - (F_{m+2} - F_m) = 3F_m - F_{m+2}.$$

Substituting this in first with $m = 2n - 3$ and then $m = 2m - 1$ gives

$$a_{n+1} = 8F_{2n-3}^2 - 8F_{2n-1}^2 + (3F_{2n-3} - F_{2n-1})^2$$

$$= (3F_{2n-1} - F_{2n-3})^2 = F_{2n+1}^2,$$

which completes the induction. □