Berkeley Math Circle: Monthly Contest 1 Solutions

1. Do there exist positive irrational numbers x and y such that $x + y$ and xy are both rational? If so, give an example; if not, explain why not.

   Solution. Such numbers do exist. One example is $x = 3 - \sqrt{2}$ and $y = 3 + \sqrt{2}$ (which are irrational due to the famous fact that $\sqrt{2}$ is irrational). Then $x + y = 6$ and $xy = 7$. □

2. Let n be an odd positive integer not divisible by 3. Show that $n^2 - 1$ is divisible by 24.

   Solution. We will show it is divisible by 8 and 3. Since the least common multiple of 8 and 3 is 24, this implies the result.

   We factor $(n - 1)^2 = (n - 1)(n + 1)$.

   To show divisibility by 8, note that $n - 1$ and $n + 1$ are two consecutive even integers. Among any two consecutive integers one of them must be divisible by 4; the other one is divisible by 2 by definition, so their product is divisible by 8.

   To show divisibility by 3, note that $\{n - 1, n, n + 1\}$ form three consecutive integers. Thus at least one of them is divisible by 3. We assumed n was not divisible by 3, so it must be either $n - 1$ or $n + 1$, hence their product is divisible by 3 as well. □

3. Four cars A, B, C, and D travel at constant speeds on the same road (not necessarily in the same direction). Car A passed B and C at 8am and 9am, respectively, and met D at 10am. Car D met B and C at 12pm and 2pm, respectively. Determine at what time B passed C. (The times given are within a single day.)

   Solution. Draw lines A, B, C, D to graph the movement of the four cars, with time on the x-axis and distance on the y-axis, and let (XY) be the point where lines X and Y meet. Then (AC) is the midpoint of the line from (AB) to (AD), and (DB) is the midpoint of the line from (DA) to (DC). Thus, (BC) is the intersection of two medians of the triangle with vertices at (AB), (AD), and (DC), so it is the centroid. But this means the distance from (AC) to (BC) is half the distance from (BC) to (CD). Since A meets C at 9am and C meets D at 2pm, the time between these meetings is 5 hours. The meeting of C and B occurs $1/3$ of the way from the first meeting to the second, i.e. $5/3$ hours after A meets C, or at 10:40am. □

4. A row of fifty coins with integer denominations is given, such that the sum of the denominations is odd. Alice and Bob alternate taking either coin at the left end of the row or the right end of the row, with Alice playing first. Prove that Alice can always ensure she gets more than half the money.

   Solution. Color the coins alternatively black and white. Since 50 is even, on Alice's turn, the coins at either end of the row are different colors.

Thus Alice could guarantee getting all of the black coins, she could also guarantee getting all of the white coins. Since either the sum of the black coins is more the sum the white coins, or vice-versa (they are not equal since the sum is odd), Alice can guarantee getting more money than Bob. □

5. Let a, b, c be positive real numbers such that $abc = 1$. Simplify

$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca}.$$

Solution. We may let $a = y/x$, $b = z/y$, $c = x/z$ for some real numbers x, y, z. Then

$$\begin{aligned}\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} &= \frac{1}{1+y/x+z/x} + \frac{1}{1+z/y+x/y} + \frac{1}{1+x/z+y/z} \\ &= \frac{x}{x+y+z} + \frac{y}{x+y+z} + \frac{z}{x+y+z} \\ &= 1.\end{aligned}$$

□

6. Two triangles ABC and XYZ have a common circumcircle. Suppose the nine-point circle γ of $\triangle ABC$ passes through the midpoints of $\overline{XY}$ and $\overline{XZ}$. Prove that γ also passes through the midpoint of $\overline{YZ}$.

Solution. Let R be the circumradius of $\triangle ABC$. Note γ has radius $R/2$ and passes through the midpoints M and N of $\overline{XY}$ and $\overline{XZ}$. There are only two circles with these properties: one of them is $\triangle XMN$ (by homothety) and the other is the nine-point circle of $\triangle XYZ$. Since γ was the nine-point circle of $\triangle ABC$, it must be the latter. □

7. Let G be a simple graph with k connected components, which have $a_1, \ldots, a_k$ vertices, respectively. Determine the number of ways to add $k-1$ edges to G to form a connected graph, in terms of the numbers a_i.

Solution. The answer is

$$a_1 \ldots a_k (a_1 + \cdots + a_k)^{k-2}$$

which generalizes Cayley's formula!

We will show that

$$f(a_1, \ldots, a_k) = k!(a_1 \ldots a_k)(a_1 + \cdots + a_k)^{k-2}$$

counts the number of ways to pick $k-1$ edges, *in order*. The proof is by induction on k, with $k = 1$ being clear. If we add an edge between the first and second connected components, there are a_1a_2 ways to do so, and the number of ways to finish is $f(a_1+a_2, a_3 \dots, a_k)$. So

$$\begin{aligned} f(a_1,\dots,a_k) &= \sum_{1\le i<j\le k} a_i a_j f(a_i+a_j, \underbrace{a_1,\dots,a_k}_{\text{missing } a_i \text{ and } a_j}) \\ &= (k-1)! \sum_{1\le i<j\le k} (a_i+a_j)(a_1\dots a_k)(a_1+\dots+a_k)^{k-2} \\ &= (k-1)!(a_1\dots a_k)(a_1+\dots+a_k)^{k-2} \sum_{1\le i<j\le k}(a_i+a_j) \\ &= k!(a_1\dots a_k)(a_1+\dots+a_k)^{k-1}. \end{aligned}$$

□