# Berkeley Math Circle: Monthly Contest 7 Solutions

1. Lisa considers the number
$$x = \frac{1}{1^1} + \frac{1}{2^2} + \cdots + \frac{1}{100^{100}}.$$
Lisa wants to know what $x$ is when rounded to the nearest integer. Help her determine its value.

   *Solution.* The answer is 1. Indeed, note that
$$x \le 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{100}}.$$
By the formula for the sum of a geometric series, we see that
$$x \le 1 + \frac{1}{2} - \frac{1}{2^{101}}.$$
Thus $x < 3/2$, and the closest integer to $x$ is 1. □

2. A number is called *cool* if it is the sum of two nonnegative perfect squares. For example the numbers 17 and 25 are cool because $17 = 4^2 + 1^2$ and $25 = 5^2 + 0^2$, but the number 15 is not cool.

   Show that if $k$ is cool, then $2k$ is cool.

   *Solution.* Observe that if $k = a^2 + b^2$, then $2k = (a-b)^2 + (a+b)^2$. □

3. Victoria paints every positive integer either pink or blue. Is it possible that both conditions below are satisfied?
   - For every positive integer $n$, the numbers $n$ and $n+5$ are different colors.
   - For every positive integer $n$, the numbers $n$ and $2n$ are different colors.

   *Solution.* The answer is no.

   Assume for contradiction that such a coloring exists. Let's say 10 was colored pink. Then $10 + 5 = 15$ must be blue, and $15 + 5 = 20$ must be pink. But now $20 = 10 \cdot 2$, violating the second condition.

   Now if 10 was colored blue, the same argument works with "pink" and "blue" switched. □

4. Let $H$ be the orthocenter of an acute triangle $ABC$. The circumcircle $\omega$ of triangle $HAB$ intersects line $BC$ at the point $D \neq B$. Let $P$ be the intersection of the line $DH$ and the line segment $AC$, and let $Q$ be the circumcenter of triangle $ADP$. Show that the center of $\omega$ lies on the circumcircle of triangle $BDQ$.

*Solution.* Reflect $D$ across $AB$ to $E$. Thus the point $A'$ diametrically opposite $A$ is the midpoint of arc $EC$. Call $P'$ the second intersection of $(ADE)$ with $AC$. Let $AA'$ meet $BC$ at $F$.

As $AD = AE = AC$, we have $\angle EAF = \measuredangle EDC$ (both $\frac{1}{2}\angle EAC$). So points $ADEF$ are cyclic. In light of this, $\measuredangle FDP' = \measuredangle FAP' = \measuredangle OAC = \measuredangle BAH = \measuredangle BDH = \measuredangle FDH$, thus $P' = P$. So $ADEFP$ is cyclic and $Q$ is its center, which naturally lies on $AB$. By reflection, we just need to show $BQOE$ is cyclic.

Since $OQ$ is the perpendicular bisector of segment $AE$, while $AH \perp BC$, and finally $\measuredangle BAH = \measuredangle EAO$, we deduce that $BQOF$ is cyclic. Finally $\measuredangle EBF = \measuredangle ECF = \measuredangle EAC = \measuredangle EOA' = \measuredangle EOF$, so pentagon $BQOFE$ is cyclic, as desired. $\square$

5. Prove that

$$64\frac{abcd+1}{(a+b+c+d)^2} \le a^2+b^2+c^2+d^2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}$$

for $a, b, c, d > 0$.

*Solution.* By Holder inequality ,we have

$$\left(\sum_{\text{cyc}} a^{-2}\right)\left(\sum_{\text{cyc}} a\right)^2 \ge (1+1+1+1)^3 = 64.$$

Also, we have

$$\left(\sum_{\text{cyc}} a^{2}\right)\left(\sum_{\text{cyc}} a\right)^2 \ge \left(4\sqrt[4]{(abcd)^2}\right)\left(4\sqrt[4]{abcd}\right)^2 = 64abcd$$

by AM-GM. Summing these two gives the conclusion. $\square$

6. Let $a_1, a_2, \dots, a_{1000}$ be real numbers such that

$$\begin{aligned}
a_1\cdot 1 + a_2\cdot 2 + a_3\cdot 3 + \cdots + a_{1000}\cdot 1000 &= 0\\
a_1\cdot 1^2 + a_2\cdot 2^2 + a_3\cdot 3^2 + \cdots + a_{1000}\cdot 1000^2 &= 0\\
a_1\cdot 1^3 + a_2\cdot 2^3 + a_3\cdot 3^3 + \cdots + a_{1000}\cdot 1000^3 &= 0\\
&\vdots\\
a_1\cdot 1^{999} + a_2\cdot 2^{999} + a_3\cdot 3^{999} + \cdots + a_{1000}\cdot 1000^{999} &= 0\\
a_1\cdot 1^{1000} + a_2\cdot 2^{1000} + a_3\cdot 3^{1000} + \cdots + a_{1000}\cdot 1000^{1000} &= 1.
\end{aligned}$$

What is the value of $a_1$?

*Solution.* The key idea is to multiply on weights which are polynomial coefficients. Specifically, let $P(n) = n(n-2)(n-3)\dots(n-1000) = \sum_{k=0}^{1000} c_k n^k$, say. Note that $c_1 = 0$ and $c_{1000} = 1$.

Now take the $k$th equation and multiply it by $c_k$, then add all equations together. From this we obtain

$$a_1P(1) + a_2P(2) + \cdots + a_{1000}P(1000) = c_{1000} = 1.$$

But $P(1) = (-1)^{999} \cdot 999!$ and $P(2) = \cdots = P(1000) = 0$. Therefore, $a_1 = \frac{-1}{999!}$. $\square$

7. Determine all primes $p$ such that there exists an integer $x$ satisfying $x^{2010} + x^{2009} + \cdots + 1 \equiv p^{2010} \pmod{p^{2011}}$.

*Solution.* The answer is all $p \equiv 1 \pmod{2011}$.

First, note that if $x$ satisfies the problem condition, then we in particular have $\Phi_{2011}(x) \equiv 0 \pmod{p}$ which implies $p \equiv 1 \pmod{2011}$.

Conversely, suppose $p \equiv 1 \pmod{2011}$ and fix an $a \pmod{p}$ with order 2011 (hence $a \not\equiv 1 \pmod{p}$). Let $A = \left\{x \pmod{p^{2011}} \mid x \equiv a \pmod{p}\right\}$ and $B = \left\{x \pmod{p^{2011}} \mid x \equiv 0 \pmod{p}\right\}$. We claim $\Phi_{2011} : A \to B$ is injective: indeed if $\Phi_{2011}(x) \equiv \Phi_{2011}(y) \pmod{p^{2011}}$ then

$$(x^{2011} - 1)(y - 1) \equiv (y^{2011} - 1)(x - 1) \pmod{p^{2011}}$$

hence

$$(x - y)\left(xy \sum_k x^{2010-k}y^k - \sum_k x^{2011-k}y^k + 1\right) \equiv 0 \pmod{p^{2011}}.$$

The sum in parentheses is nonzero modulo $p$ because it equals

$$\begin{aligned} a^2 \cdot 2011a^{2010} - 2012a^{2011} + 1 &= 2011a^{2012} - 2012a^{2011} + 1 \\ &\equiv 2011a - 2012 + 1 \pmod{p} \\ &= 2011(a - 1) \\ &\not\equiv 0 \pmod{p}. \end{aligned}$$

This proves $\Phi_{2011} : A \to B$ is injective; but now $|A| = |B|$ implies surjectivity too. Since $p^{2010} \in B$, this completes the proof. $\square$