Berkeley Math Circle: Monthly Contest 4 Solutions

1. On an 6×6 chessboard, we randomly place counters on three different squares. What is the probability that no two counters are in the same row or column?

   Solution. The number of ways to pick three squares is $\binom{36}{3}$.

   We now count the number of ways to pick three squares with no two in the same row or column. One can select three distinct rows, in $\binom{6}{3}$ ways, that we will place the counters in. Afterwards, there are 6 ways to pick the column for the first row, 5 to pick the column in the second row, and 4 for the final row.

   So the answer is
$$\frac{\binom{6}{3} \cdot 6 \cdot 5 \cdot 4}{\binom{36}{3}} = \frac{20 \cdot 6 \cdot 5 \cdot 4}{36 \cdot 35 \cdot 34/6} = \frac{40}{119}.$$
□

2. Alice picks an *odd* integer n and writes the fraction
$$\frac{2n+2}{3n+2}.$$
   Show that this fraction is already in lowest terms. (For example, if $n = 5$ this is the fraction $\frac{12}{17}$.)

   Solution. Let $A = 2n + 2$ and $B = 3n + 2$. Now notice that
$$3A - 2B = 3(2n+2) - 3(3n+2) = 2.$$
   So if some integer $d \geq 1$ divides both A and B, it also divides $3A - 2B = 2$. Hence d must be 1 or 2.

   But since n was odd, the number $3n + 2$ is odd, and so we can't have $d = 2$. Thus the only common divisor of A and B is 2, as needed. □

3. Let ABC be a triangle. A line is drawn not passing through any vertex of ABC. Prove that some side of ABC is not cut by the line.

   Solution. Consider the two sides of the line ℓ. By pigeonhole principle on the three vertices of ABC, two of these verctices, say A and B, lie on the same side of ℓ. Then segment AB does not intersect ℓ.. □

4. A sequence $a_1, a_2, \ldots$ of positive integers satisfies
$$a_{n+1} = a_n^3 + 103$$
   for every positive integer n. Prove that the sequence contains at most one perfect square.

Solution. It's easy to check that no two consecutive terms can be perfect squares, since the only squares which differ by 103 are 51^2 and 52^2.

Now, note that squares are 0, 1, or 4 mod 8. After a perfect square appears, the next term must be -1 or 0 mod 8, and thereafter all terms are -1, -2 modulo 8, so no more squares. $\square$

5. Show that n divides $\varphi(a^n - 1)$ for any integers a and n, where φ is Euler's totient function.

Solution. Let $N = a^n - 1$. Then $\gcd(a, N) = 1$ and the order of $a \pmod N$ is exactly equal to n. But $a^{\varphi(N)} \equiv 1 \pmod N$ too. Thus n divides $\varphi(N)$. $\square$

6. Let a, b, c be pairwise distinct integers. Prove that

$$\frac{a^3 + b^3 + c^3}{3} \geq abc + \sqrt{3(ab + bc + ca + 1)}.$$

Solution. Let $3k^2 - 1 = ab + bc + ca$, so we need $a^3 + b^3 + c^3 \geq 3(abc + 3k)$. Now, we have

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 2^2 + 1^2 + 1^2 = 6.$$

In particular, we get

$$(a+b+c)^2 = \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] + 3(3k^2 - 1) \geq 9k^2.$$

Thus $a + b + c \geq 3k$. Now, using the factorization of $a^3 + b^3 + c^3 - 3abc$ gives

$$a^3 + b^3 + c^3 - 3abc \geq (3k)(\frac{1}{2} \cdot 6) = 9k$$

as desired. $\square$

7. Let $AXYZB$ be a convex pentagon inscribed in a semicircle with diameter $\overline{AB}$, and let K be the foot of the altitude from Y to $\overline{AB}$. Let O denote the midpoint of $\overline{AB}$ and L the intersection of $\overline{XZ}$ with $\overline{YO}$. Select a point M on line KL with $MA = MB$, and finally, let I be the reflection of O across $\overline{XZ}$. Prove that if quadrilateral $XKOZ$ is cyclic then so is quadrilateral $YOMI$.

Solution. Extend the semicircle to a circle Γ. Let line LK meet Γ again at two points P and Q. Let W be the point on ray OM such that $OW \cdot OM = OA \cdot OB$. So points P, Q, W, O are concyclic, say on γ.

Now, L is the radical center of γ, Γ, and the circumcircles of $XKOZ$, because lines XZ and PQ are radical axii. So, line YO is the radical axis of Γ and γ.

Let T denote the intersection of lines XZ and AB. We have that $KO \cdot KT = KA \cdot KB = KP \cdot KQ$, so point T also lies on γ. Also, according to $TA \cdot TB = TK \cdot TZ = TK \cdot TO$, we deduce that $\overline{TY}$ is tangent to Γ.

Finally, let S denote the midpoint of $\overline{YW}$. By a homothety of ratio 2 at W, we have that the line passing through S and the midpoint of $\overline{WT}$ is perpendicular to

$\overline{YO}$. Moreover, S lies on the perpendicular bisector of line $\overline{KO}$. Therefore, S is the center of the circumcircle of quadrilateral $XKOZ$.

Finally, the collinearity of Y, S, W implies quadrilateral $YOMI$ is concyclic, since one can readily show that $OS \cdot OI = OW \cdot OM = OY^2$, hence $\angle OIM = \angle OWS = \angle OWY = \angle OYM$. $\square$