Berkeley Math Circle: Monthly Contest 2 Solutions

1. A rectangle $\mathcal{R}$ has perimeter is $2\sqrt{2018}$ and the diagonal has length $\sqrt{2017}$. What is its area?

   Solution. Let a and b be the side lengths, so

   $$a + b = \sqrt{2018}$$
   $$a^2 + b^2 = 2017$$

   the last by the Pythagorean Theorem. Squaring the first relation gives $(a+b)^2 = a^2 + b^2 + 2ab = 2018$, hence $2ab = 1$, and the area of the rectangle is $ab = \frac{1}{2}$. $\square$

2. For odd positive integers a, b, c prove that

   $$a^4 + b^4 + 2017 \neq c^4.$$

   Solution. Note that the last digit of n^4, where n is an odd positive integer, is always either 1 or 5. But $1 + 1 + 7$ ends with 9, $1 + 5 + 7$ ends with 3 and $5 + 5 + 7$ ends with 7. So the last digits of $a^4 + b^4 + 2017$ and c^4 must be different, which implies they are not equal. $\square$

3. In quadrilateral $ABCD$ we have $AB = 7$, $BC = 24$, $CD = 15$, $DA = 20$, and $AC = 25$. What is the length of BD?

   Solution. Observing $7^2 + 24^2 = 15^2 + 20^2 = 25^2$, we conclude that triangle ABC and ADC are right. So in particular quadrilateral $ABCD$ can be inscribed in a circle with diameter AC. Then by Ptolemy's Theorem we get

   $$BD = \frac{7 \cdot 15 + 20 \cdot 24}{25} = \frac{580}{25} = \frac{116}{5}. \qquad \square$$

4. A country has 50 states. How many ways are there to join some pairs of them by two-way flights such that every state has an odd number of flights departing it?

   Solution. The answer is $2^{\binom{49}{2}}$. Rather than 50 states, we will consider the nation as having 49 states and a 50th *capital.* The claim is that we can actually join 49 of the states in any way we wish, and there will be a unique way to join the capital to the remaining 49 states.

   Suppose we've joined the first 49 states $S_1, \ldots, S_{49}$. Some of these states have an odd number of flights, and we call these *odd states*, The others have an even number of flights, and we call these *even states.*

   In light of this there really is *at most* one way to join the final capital C: namely, we must link C to all the even states, but none of the odd states. However C itself needs to have an odd number of flights departing it — so what we have to show is that the number of even states is odd. Since 49 is odd, this is the same as checking the number of odd states is even.

This is actually a classical problem sometimes called the "handshake lemma". Call the *degree* of a state the number of flights serving it (not including the one from the capital if it exists). By double-counting, the sum of the degrees is exactly twice the number of flights (since each flight increase the degree of exactly two states by 1). So the sum of the degrees is even, which means the number of odd degrees is even, as desired. $\square$

5. Determine whether there exist polynomials $A(x)$, $B(x)$, $P(y)$, $Q(y)$ with real coefficients satisfying
$$x + y + (xy)^{2017} = A(x)P(y) + B(x)Q(y).$$

Solution. The answer is no. Suppose we actually select $y \in \{-1, 0, 1\}$; we then get the equations

$$\begin{aligned} 1 + x + x^{2017} &= \lambda_1 A(x) + \mu_1 B(x) \\ x &= \lambda_2 A(x) + \mu_2 B(x) \\ -1 &= \lambda_3 A(x) + \mu_3 B(x) \end{aligned}$$

for six real numbers λ_1, λ_2, λ_3.

We claim this is impossible. The last two equations can be solved to give

$$\begin{aligned} (\mu_3\lambda_2 - \mu_2\lambda_3)A(x) &= \mu_3 x + \mu_2 \\ (\mu_2\lambda_3 - \mu_3\lambda_2)B(x) &= \lambda_3 x + \lambda_2 \end{aligned}$$

which implies in particular that $\deg A, \deg B \le 1$. So the first equation is certainly not possible. $\square$

6. Solve the equation $a^2 + b^2 + c^2 = (ab)^2$ over the integers.

Solution. The answer is $(0, 0, 0)$ which is seen to work.

Henceforth assume $a, b, c \ge 0$, since we may replace a with $-a$. The given can be rewritten as
$$c^2 + 1 = (a^2 - 1)(b^2 - 1).$$

If $\min(a, b) \le 1$ we easily see only solution is $(a, b) = (0, 0)$. Hence in the sequel assume $a, b \ge 2$.

First note that we cannot have a or b odd, since c^2+1 is never divisible by 4. So a and b are even and both sides are odd integers. But now the right-hand side is the product of two 3 (mod 4) factors. On the other hand, by Fermat's Christmas theorem, we know that $c^2 + 1$ only has 1 (mod 4) prime factors. This is a contradiction. $\square$

7. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a cyclic octagon. Let B_i by the intersection of A_iA_{i+1} and $A_{i+3}A_{i+4}$ (where indices are taken modulo 8). Prove that $B_1, B_2, \dots, B_8$ lie on a conic.

Solution. Consider the hexagon

$$B_2B_5B_8B_3B_6B_1.$$

The sides B_2B_5, B_5B_8, B_8B_3, B_3B_6, B_6B_1 coincide with the lines A_5A_6, A_8A_1, A_3A_4, A_6A_7, A_1A_2, respectively, by definition. Consequently,

- $B_2B_5 \cap B_3B_6 = A_5A_6 \cap A_6A_7 = A_6$.
- $B_5B_8 \cap B_6B_1 = A_8A_1 \cap A_1A_2 = A_1$.
- $B_8B_3 \cap B_1B_2 = A_3A_4 \cap B_1B_2$.

But by Pascal's Theorem on the hexagon $A_0A_2 \dots A_6$, the three points B_1, B_2, $A_1A_6 \cap A_3A_4$ are collinear. Equivalently, $A_3A_4 \cap B_1B_2$ lies on line A_1A_6.

Thus by the converse of Pascal's Theorem on $B_1B_6B_3B_8B_5B_2$ this implies that the six points B_i $(i \neq 4, 7)$ lie on a conic. A suitable cyclic permutation of indices, combined with the fact that five points determine a unique conic, solves the problem. $\square$