Berkeley Math Circle: Monthly Contest 1 Solutions

1. Rachel measures the angles of a certain pentagon $ABCDE$ in degrees. She finds that $\angle A < \angle B < \angle C < \angle D < \angle E$, and also that the angle measures form an *arithmetic progression*, meaning that $\angle B - \angle A = \angle C - \angle B = \angle D - \angle C = \angle E - \angle D$.

   What was the measure of $\angle C$?

   Solution. The answer is 108 degrees. Indeed, in a pentagon the sum of the angles is $180 \cdot 3 = 540$ degrees. Also, in an arithmetic progression the middle term is the average, which is $\frac{1}{5} \cdot 540 = 108$. □

2. Victor has four red socks, two blue socks, and two green socks in a drawer. He randomly picks two of the socks from the drawer, and is happy to see that they are a matching pair. What is the probability the pair was red?

   Solution. There are $\binom{4}{2} = 6$ possible red pairs, and then $\binom{2}{2} = 1$ blue pairs, and finally $\binom{2}{2} = 1$ green pairs.

   So there were a total of $6 + 1 + 1 = 8$ possible pairs, of which there were 6 red ones. So the answer is $3/4$. □

3. Kelvin the frog jumps along the number line starting at 0. Every time he jumps, he jumps either one unit left or one unit right. For example, one sequence of jumps might be $0 \to -1 \to 0 \to 1 \to 2 \to 3 \to 2$.

   How many ways are there for Kelvin to make exactly 10 jumps and land on a prime number? (The prime numbers are the sequence $2, 3, 5, 7, \dots$. Negative numbers are not considered prime.)

   Solution. First, note that every time Kelvin jumps, he must jump from an even number to an odd one or vice-versa. Thus after ten jumps, he must land on an even number. So, if that number is to be prime, it must be 2.

   This means Kelvin must make 6 jumps right and 4 jumps left. That means the answer is $\binom{10}{6} = 210$. □

4. Decide whether that there exists an infinite set S of positive integers with the property that if we take any finite subset T of S, the sum of the elements of T is not a perfect kth power for any $k \geq 2$.

   Solution. Consider the set

   $$S = \left\{2,\ 2^2 \cdot 3,\ 2^2 \cdot 3^2 \cdot 5,\ 2^2 \cdot 3^2 \cdot 5^2 \cdot 7,\ 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11, \dots\right\}.$$

   Let $T \subseteq S$ be a finite nonempty set, and m its smallest element. Let p be the prime factor which appears exactly once in m. Then p divides every element of T, and p^2 divides every element of T except m. From this it follows that the sum of the elements of T has exactly one factor of p; thus, it cannot be a kth power. □

5. Solve for real x:

$$x+\sqrt{(x+1)(x+2)}+\sqrt{(x+2)(x+3)}+\sqrt{(x+3)(x+1)}=4.$$

Solution. First, note by monotonicity of LHS there is exactly one solution.

Let $z=x+2$ and add 2 to both sides. Then we obtain

$$\left(\sqrt{z}+\sqrt{z+1}\right)\left(\sqrt{z}+\sqrt{z-1}\right)=6.$$

Then, we can multiply through to get

$$\begin{aligned}\sqrt{z}+\sqrt{z-1}&=6\left(\sqrt{z+1}-\sqrt{z}\right)\\ 7\sqrt{z}&=6\sqrt{z+1}-\sqrt{z-1}\\ 49z&=36(z+1)+(z-1)-12\sqrt{z^2-1}\\ 12z&=35-12\sqrt{z^2-1}\\ (12z-35)^2&=12^2(z^2-1)\\ 35^2+12^2&=2\cdot 12\cdot 35z.\end{aligned}$$

Solving gives $z=\frac{1369}{840}$, hence $x=-\frac{311}{840}$. □

6. On a circle we write $2n$ real numbers with a positive sum. For each number, there are two sets of n numbers such that this number is on the end. Prove that at least one of the numbers has a positive sum for both these sets.

Solution. By decreasing each number by the average of all n numbers, we may assume for convenience that the sum is in fact 0. Let X_k denote the contiguous sequence of numbers in positions $k, k+1, ..., k+(n-1)$ (modulo $2n$). Remark that X_i and X_j share an endpoint if and only if $i-j\in\{n-1,n+1\} \pmod{2n}$. Furthermore, because the sum is all numbers is 0, we have $X_i=-X_{i+n}$. In particular, it suffices to show two sequences sharing an endpoint have the same sign (not necessarily positive).

Assume this is not the case. Construct a graph G whose vertices are the residues modulo $2n$, joining i to j iff $i-j\in\{n-1,n,n+1\} \pmod{2n}$.

Now color i red if $X_i\geq 0$ and blue otherwise. By the hypothesis, this is a 2-coloring of G. But this is impossible since G is not bipartite! For odd n there is an odd cycle $0, n-1, 2(n-1), \cdots, n(n-1)\equiv 0$ of length n, and for even n we can also construct the odd cycle $0, n-1, 2(n-1), \cdots, n(n-1)\equiv n, 0$, which has length $n+1$. □

7. Let $ABCD$ be a convex quadrilateral. Assume that the incircle of triangle ABD is tangent to $\overline{AB}$, $\overline{AD}$, $\overline{BD}$ at points W, Z, K. Also assume that the incircle of triangle CBD is tangent to $\overline{CB}$, $\overline{CD}$, $\overline{BD}$ at points X, Y, K. Prove that quadrilateral $WXYZ$ is cyclic.

Solution. From the concurrence of the Gregonne point, it follows that lines WZ, XY, and BD concur at the harmonic conjugate T of K with respect to $\overline{BC}$. (One can also see the concurrence directly by applying Ceva and Menelaus.) Then $TK^2 = TW\cdot TZ = TX\cdot TY$, as desired. □