Berkeley Math Circle: Monthly Contest 6

Due April 9, 2025

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.
- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 6, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, April 9, 2025 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.
- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus – no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.
- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 6

1. How many distinct products can be formed by multiplying together one or more different elements of the set $\{1, 2, 3, 4, 8\}$?

   For example, the product $2 \cdot 3$ is equivalent to the product $1 \cdot 2 \cdot 3$ and so should both only be counted as one distinct product, not two. Equivalently, the ordering of the multiplication should not be considered. For instance, the products $1 \cdot 2 \cdot 3$ and $2 \cdot 3 \cdot 1$ are equivalent.

2. Let $ABCD$ be a convex quadrilateral with $AD = 25$, $AB = 7$, $BD = 24$, $AC = 20$, and $CD = 15$. Let $EFGH$ be an isosceles trapezoid formed by connecting three equilateral triangles of side length $\frac{25}{\sqrt{3}}$. Which has larger area, quadrilateral $ABCD$ or quadrilateral $EFGH$?

3. A stack of n cards is labeled with the integers from 1 to n such that the kth bottom-most card is labeled with k. Thus the topmost card is labeled with n, the second topmost card is labeled with $n-1$, all the way to the bottom card labeled with 1.

   Aerith has two boxes, a red box and a blue box. At all points in time, the *score* of a box is the sum of the numbers written on the cards it contains. Every minute, Aerith draws the topmost card from the stack and inserts it in whichever box currently has the lower score, breaking ties arbitrarily. Let R and B respectively be the scores of the red and blue boxes at the very end. Prove that $R - B$ is either -1, 0, or 1.

4. For any integer k, let $\rho(k)$ denote the set of prime numbers that divide k. As an example, note that $\rho(35) = \{5, 7\}$ and $\rho(36) = \{2, 3\}$.

   Determine whether or not there exist infinitely many pairs of positive integers (m, n) that simultaneously satisfy the equalities $\rho(m) = \rho(n)$ and $\rho(m+1) = \rho(n+1)$.

5. Let n be a positive integer. What is the value of

$$\sum_{k=0}^{n-1} (-1)^k k^{n-1} \binom{n}{k}?$$

6. Does there exist a two-player game, played only with a fair coin, such that
   - the probability that the game will never terminate is zero, and
   - the probability of one of the players winning is an irrational number?

7. Fixed positive rational numbers p and q are chosen such that both $2\sqrt{p}$ and $q - p$ are integers. Let $\alpha = \sqrt{p} + \sqrt{q}$, and define the set S such that it contains exactly all numbers of the form $m\alpha + n$, where m and n are any integers. Given that α is not an integer, show that the set $S \cap (0, r)$ is nonempty for all positive real numbers r.