Berkeley Math Circle: Monthly Contest 5

Due March 12, 2025

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 5, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, March 12, 2025 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus – no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 5

1. Let $ABCD$ be a quadrilateral. Let ω_{ABC} be the circle passing through the points A, B, and C. Define the circles ω_{ABD}, ω_{ACD}, and ω_{BCD} similarly. Suppose that the circles ω_{ABC}, ω_{ABD}, ω_{ACD}, and ω_{BCD} all have the same radius. Must they all be the same circle?

2. The residents of Metropolis are back at it again! Recall that Metropolis is an infinite city built on a rectangular coordinate system. This time, for all whole numbers x and y, the Metropolians have installed a cell tower of height $101x^2 + 10y^2$ at the location (x, y). Determine the number of cell towers of height 33181, and find where they are located.

3. Consider an infinite positive integer sequence $n_1, n_2, n_3, \ldots$ defined such that $n_1 = 4$, and for all $k \geq 2$ the value of n_k is generated from the previous one, n_{k-1}, as follows.
   - Let $m_{k-1} = n_{k-1}^3$.
   - Let s_{k-1} be the sum of the digits of m_{k-1}.
   - Let n_k be the remainder of the $s_{k-1} + 5$ upon division by 100.

   For example, for $k = 2$, observe that $n_1 = 4$. Then $m_1 = 64$ and $s_1 = 10$, so $n_2 = 15$.

   Find n_{2025}.

4. What are the last two digits of the decimal expansion of $\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{2024\text{ 2's}}$?

5. During her summer vacation, Aerith took 32 different photos. Can she create 7 albums, each containing exactly 12 photos from her vacation, so that every pair of albums has at most 3 photos in common?

6. Show that there are exist an infinite number of ordered pairs of integers (m, n) satisfying $|m^2 - 2n^2| = 1$.

7. Let ABC be an acute triangle and let P be a point inside. Let the projection of P onto BC, CA, and AB be D, E, and F, respectively. Show that the value of $BD \cdot BC + CE \cdot CA + AF \cdot AB$ does not depend on the choice of P.