Berkeley Math Circle: Monthly Contest 7

Due April 24, 2024

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 7, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, April 24, 2024 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 7

1. The *factorial* of a positive integer n, denoted $n!$, is the product of all positive whole numbers less than or equal to n. For example, $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$.

   The *double-factorial* of a positive integer n, denoted $n!!$, is the product of all *even* positive whole numbers less than or equal to n. For instance, $7!! = 6!! = 2 \cdot 4 \cdot 6$.

   Which is greater, $(2024!!)!$ or $(2024!)!!$?

2. There is a curvy loop in the plane, with no straight line intersecting it at more than two points. Prove that any point on the curvy loop is a vertex of an equilateral triangle whose vertices all lie on the curvy loop.

3. Find all functions f from the integers to the integers such that there exists another function g, also from the integers to the integers, such that $g(0) \neq 0$ and

   $$f(xg(y) + z) = f(yg(x) + z)$$

   for any integers x, y, and z.

4. Find all ordered triplets of integers (x, y, z) satisfying $z^2 = x^2 + xy$ and $y^2 = z^2 + xz$.

5. Let n be some positive integer such that $2n + 5$ divides either $5n! + 2$ or $5n! - 2$. Find all possible values of n.

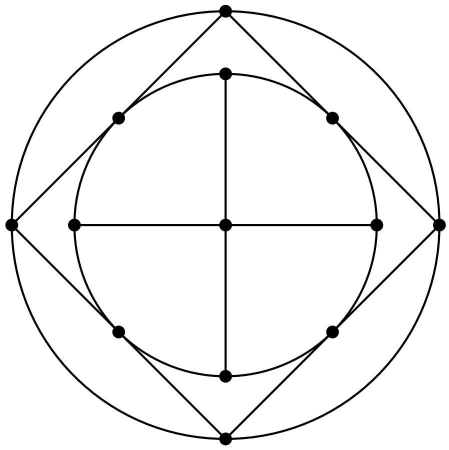
6. The creatures of Mathland believe in the following creation myth.

   - In the beginning, there was nothing, not even space.
   - Then came the v *vertices*, a finite collection of points.
   - Then came the e *edges*, each created by an imaginary path connecting two edges.
   - Then came the f *faces*, each created by an imaginary film attached to a circle of edges.
   - Then came the c *cells*, each created by an imaginary solid whose boundary is a set of faces that are attached to a polyhedron of polygons.
   - A *world* is the set of all vertices that can be reached from a starting vertex by means of edges.
   - A *loop* is a (potentially empty) set of edges that uses each vertex an even number of times. Two loops are considered the same if the set of edges where they disagree forms the boundary of a set of faces.
   - A *creature* is a (potentially empty) set of faces that uses each edge an even number of times. Two creatures are considered the same if the set of faces where they disagree forms the boundary of a cell.
   - A *god* is a (potentially empty) set of cells that uses each face an even number of times.

   Let κ, λ, γ, and ω respectively denote the number of creatures, loops, gods, and worlds. Prove that

   $$\frac{\kappa}{\lambda\gamma} = 2^{v-e+f-c-\omega}.$$

7. Aerith randomly erases some of the 24 edges in the following diagram, such that each edge is erased independently with probability $0.5 - p$ for some fixed $p \in [-0.5, 0.5]$. Prove that the expected number of bounded regions into which the plane is divided by the new figure is given by $12p + 12f(p)$, where f is an odd Lipshitz polynomial.



Note: A function g is said to be *Lipshitz* if it satisfies $|g(x) - g(y)| \leq |x - y|$ for all x and y in its domain.