Berkeley Math Circle: Monthly Contest 6

Due March 27, 2024

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 6, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, March 27, 2024 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 6

1. Is the number $23^{24} - 24^{23}$ positive or negative?

2. V. Enhance, the CEO of Evan Corporation LLC, has a secret favorite number c, not necessarily a whole number. He also has a special number machine with a red button: when any number x is inputted into the machine and the button is pressed, it displays the value of
$$\frac{x}{2} + \frac{c}{2x}$$
on a screen.

   Aerith begins by inputting the value $x = 1$ into the machine; every minute, she pushes the red button and reenters the value displayed on the screen into the machine. Interestingly, Aerith notices that output of the machine eventually gets closer and closer to exactly 206. Determine the value of c.

3. A finite collection of circles, of any radius, in the plane are shaded blue. Your task is to shade some of the circles red, while ensuring that no red circles overlap. Prove that you can always ensure that the area of the red region is at least 10% of the area of the blue region.

4. Aerith and Bob play the following game: a positive integer n is chosen, after which Aerith and Bob alternate choosing an integer between 1 and n, inclusive, that has not been chosen. They keep a running product of all numbers that have already been chosen, and the first player to make that running product a multiple of n loses. Find all initial choices of n for which Aerith wins.

5. Find the number of polynomials $P(x)$ of degree 3 with nonnegative integer coefficients strictly less than 100 such that the last two digits of $P(x)$ are either 00 or 76 for all integer values of x.

6. A line is drawn in the plane. You have a straightedge, but no compass. Prove that it is impossible to construct a parallel line.

7. Let f and g be polynomials in x, y with integer coefficients.

   (a) Prove that if some integer s is expressible as a product of coefficients of f and g, there exists a positive integer n such that s^n is expressible as an integer linear combination of the coefficients of fg.

   (b) Solve part (a) where f and g are multivariable polynomials instead.

   Note: an *integer linear combination* of a set of integers $\{a_1, \cdots, a_n\}$ is a number of the form $a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$ where $\{b_1, \cdots, b_n\}$ is an arbitrary set of integers.