Berkeley Math Circle: Monthly Contest 5

Due February 28, 2024

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 5, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, February 28, 2024 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 5

1. Define the function f on the positive integers such that

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even,} \\ 5x+1 & \text{if } x \text{ is odd.} \end{cases}$$

Find the smallest positive integer n for which there does not exist some positive integer m such that $f^m(n) = 1$. (In other words, we want the smallest n such that $f(n) \neq 1$, $f(f(n)) \neq 1$, $f(f(f(n))) \neq 1$, $f(f(f(f(n)))) \neq 1$, and so on.)

2. Completely factor $N = 2^{30} - 1$ into prime numbers.

3. Jessica owns four pairs of socks, which come in four different colors: ultramarine, vermilion, wisteria, and xanthous. Each Wednesday, she does her laundry, and every time exactly one sock goes missing.

   (a) List all possible color combinations for her four pairs of socks. Here, two color combinations are considered the same if the number of pairs of socks of the same color match; for example, having three pairs of ultramarine socks and one pair of vermilion socks and having three pairs of vermilion socks and one pair of wisteria socks are considered the same.

   (b) For each of the color combinations above, calculate the time, in weeks, until she runs out of socks to wear. Assume that she wears only matching socks at any given time, so that if she has just one sock of a given color, she throws that useless sock away.

4. Six mathematicians stand around a tree in a circle. Each has a hat whose color is randomly either red or blue. They cannot see the color of their own hat, nor the color of the hat of the person across from them, but they can see the hats of the four other mathematicians. If they can pick their strategy beforehand, what is the maximum chance they could have for everyone silently guessing their hat color right?

5. Fix some positive integer n, and consider all convex polygons P with n sides. For each such P, draw n circles, with each side being a diameter. Prove that there exists some polygon P and point X in the interior of P, with X uncovered by the n circles, iff $n \geq 5$.

6. Prove that $\tan(1^\circ)$ is an irrational number.

7. Let a_1, a_2, a_3, b_1, b_2, b_3 be arbitrary positive numbers. Prove that

$$(a_1^7 + a_2^7 + a_3^7)(b_1^7 + b_2^7 + b_3^7) \geq (a_1^4 b_1^3 + a_2^4 b_2^3 + a_3^4 b_3^3)(a_1^3 b_1^4 + a_2^3 b_2^4 + a_3^3 b_3^4).$$