Berkeley Math Circle: Monthly Contest 4

Due January 22, 2025

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 4, Problem 2
Evan O'Dorney, BMC Beginners I
Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, January 22, 2025 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus – no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 4

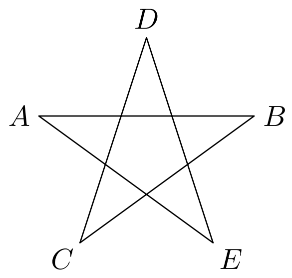
1. Among all positive whole numbers from 1 to 2024, are there more numbers that are divisible by 5 but not by 6, or are there more numbers that are divisible by 6 but not by 5?

2. There is a spinner, divided into n unequal sectors, each labeled with the numbers 1 through n. For all integer values of k with $1 \le k \le n$, the spinner lands on the number k with a probability that is exactly k^2 times the probability that it lands on the number 1. Also, it turns out that there exists some number m such that the probability of the spinner landing on a number less than m is exactly $\frac{1}{11}$. What is the smallest possible value of n?

3. Let $ABCDE$ be a regular square pyramid with square base $ABCD$ such that $AB = 1$ and $AE = \frac{\sqrt{5}}{2}$. Let F be the midpoint of AB and let G be the center of the square $ABCD$. Let H lie on EG such that FH bisects $\angle EFG$. Find GH.

4. Let $n \ge 3$ be a positive integer. Consider polynomials of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$ for which none of a_n, a_{n-1}, and a_1 are integers. Determine, with proof, whether it is possible for $P(m)$ to be an integer whenever m is an integer.

5. Observe that the five-pointed star is a self-intersecting pentagon such that each segment crosses exactly two other segments. For instance, in the following diagram, segment AB crosses segments CD and DE.



   Is it possible to draw a self-intersecting pentagon such that each segment crosses exactly one other segment? What about a self-intersecting 20021-gon?

6. A function $a(x, y)$ is said to be an *increasing integer partition* if, for all positive integers k, the sequence $(a(k,1), a(k,2), a(k,3), \dots)$ is increasing, and if, for all positive integers n, there exists exactly one pair of positive integers (i, j) such that $a(i, j) = n$. Is it possible that there exists some increasing integer partition $a(x, y)$ for which $a(x, y) \le f(x + y)$, for all positive integers x and y, where
   (a) $f(x) = x^2$?
   (b) $f(x) = Cx^{1.5}$ for some constant C?
   (c) $f(x) = Cx^{1.01}$ for some constant C?

7. Solve over the real numbers the system of equations

$$ca(a-8) = a(3a-59) + (2-1)(83-16c),$$
$$ab(b-12) = b(b-33) + (3-1)(39-18a),$$
$$bc(c-10) = c(2c-35) + (2+3)(19-5b).$$