Berkeley Math Circle: Monthly Contest 3

Due December 4, 2024

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 3, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, December 4, 2024 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus – no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

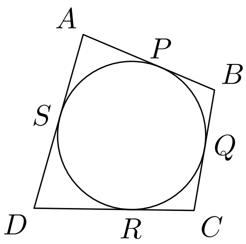
- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 3

1. Let $ABCD$ be a quadrilateral. There is a circle tangent to the sides AB, BC, CD, and DA, as shown. Show that $AB + CD = AD + BC$.



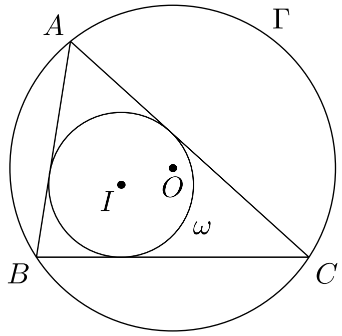
2. The infinite city of Metropolis is built on a rectangular coordinate system. For all whole numbers x and y, the Metropolians have built a building of height $xy - 6x - 4y$ at the location (x, y). (Buildings of negative height are built as underground cellars.) How many buildings have a height of 2024?

3. For which integers n is $(n^2 + n)^2 + 3$ a prime number?

4. A positive integer n is said to be *magnificent* if, for some $k > 1$, it is possible to write n as the product of k positive integers as well as the sum of the exact same k positive integers. For example, note that 6 is magnificent as $6 = 3 + 2 + 1 = 3 \cdot 2 \cdot 1$. This construction uses $n = 6$ and $k = 3$.

   Is 99 a magnificent number? What about 101?

5. Circles Γ and ω are drawn in the plane with ω wholly contained inside of Γ. Distinct points A, B, and C are chosen on Γ such that the segments AB, BC, and CA are all tangent to ω, as shown. Let Γ have center O and radius R, and let ω have center I and radius r. Prove that $OI^2 + 2Rr = R^2$.



6. A small company that provides medical services has an ambulance and five employees, Aerith, Bob, Cantor, Perelman and Landau. When they receive a call for services, they send the ambulance with a team of three employees. It turned out that at the end of a day Aerith made five trips, which is more than anyone else, while Bob made two trips, which is fewer than anyone else. How many trips did the ambulance make on that day?

7. What is the value of

$$\sum_{k=1}^{365} \frac{365! \cdot k}{(365-k)! \cdot 365^k}?$$