Berkeley Math Circle: Monthly Contest 2

Due October 23, 2024

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 2, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, October 23, 2024 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus – no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 2

1. Bob has had a bad experience with the laundry machine and now has three black socks, three blue socks, and three red socks in his sock drawer. He draws three socks from the drawer, one at a time, without placing any sock that had been drawn back in the drawer. What is the probability that at least two of these three socks share the same color?

2. Certain positive integers n have the property that, for all positive even numbers m, the last two digits of mn and the last two digits of m are exactly the same. In fact, there are exactly two positive integers less than 100 with this property. What are they?

   Note that numbers below 10 are treated as having two digits. For example, the last two digits of 8 are said to be 08.

3. Define points $A = (0,0)$, $B = (0,5)$, $C = (3,0)$, and $D = (3,5)$ on the coordinate plane. How many circles of radius 1 can be drawn entirely within rectangle $ABDC$ without overlapping?

4. A straight line segment of length 1 is given in the plane. Draw a line segment of length $\sqrt{\sqrt{5}-2}$ using only a compass and a straightedge.

5. Find all nonzero polynomials $P(x)$ satisfying $x^3P(P(x^3)) = P(x^4P(x^2))$ for all real numbers x.

6. A *multiset* is an unordered set where an element of the set can appear multiple times. We call a multiset of integers, all strictly greater than 1, *a division partition of x* if the elements of the multiset multiply to x. For any positive integer n, let $f(n)$ be the number of distinct division partitions of n. For instance, the multiset $\{2,2,2,3,30\}$ is a division partition of 720, and one can check that $f(100) = 9$. Show that $f(10^{10}) + f(40^5) + f(250^5) < 2.5^{19}$.

7. Prove that there exist two real numbers $a, b > 1$ such that there are infinitely many pairs of positive integers m, n such that $0 < a^m - b^n < 1$.