Berkeley Math Circle: Monthly Contest 1

Due September 25, 2024

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 1, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, September 25, 2024 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus – no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 1

1. How many positive integer factors of 10^{10} are not factors of 8^{10}? For example, 1, 2, 3, 4, 6, and 12 are all positive integer factors of 12, but 7 and 9 are not. Note that both 1 and 10^{10} count as valid factors of 10^{10}.

2. Four distinct points A, B, C, and D lie on a plane. Can it be that the distances AB, AC, AD, BC, BD, and CD, in some order, are
   (a) 1, 1, 1, 1, 1, 2?
   (b) 1, 2, 3, 4, 5, 6?

3. A castle has infinitely many rooms labeled $1, 2, 3, \dots$, which are divided into some number of halls. It is known that room n is on the same hall as rooms $3n + 1$ and $n + 81$ for every n. At most how many halls can this castle have?

4. Which is bigger, $A = 1^{100} + 2^{100} + \cdots + 99^{100}$, or $B = 100^{100}$?

5. There are 100 bags of money, numbered from 1 to 100. The bag with number n contains \$$n$ for all n. Then, 100 people are asked one by one which bag they would like to get money from, and can pick based on the responses of everyone who came before. Once an answer has been collected from everyone, each bag will be split equally among the people who picked it. If they are all playing optimally so as to maximize their individual profit, how many bags will not be picked?

6. Let $C(p)$ be an annual calendar for year p. Two calendars for years p and q are said to be *identical* if every date on both calendars falls on the same day of the week, and we denote this as $C(p) = C(q)$. For example, $C(2021)$ and $C(2027)$ are identical, because January 1 on both calendars falls on a Friday, and both years are non-leap years. Call the smallest positive integer N for which $C(p) = C(p + N)$ for all p the *cicada period* of the calendar system C.
   (a) Is the Julian calendar system C_J periodic? If so, what is its cicada period? In the Julian calendar, a year is a leap year if and only if it is divisible by four.
   (b) Repeat the above part for the Gregorian calendar system C_G. It features a more elaborate and accurate definition of leap years, in which p is a leap year if and only if either p is divisible by four but not by 100, or that p is divisible by 400. For example, the years 1700, 1800, and 1900 are not leap years, but the years 1600 and 2000 are.

7. Let a, b, c be positive real numbers such that $a + b + c = \frac{3}{2}$. Show that

$$\frac{a}{a(2a^2+1)+b+c} + \frac{b}{b(2b^2+1)+c+a} + \frac{c}{c(2c^2+1)+a+b} \le \frac{6}{7}.$$