

Berkeley Math Circle: Monthly Contest 8

Due Apr 26, 2023

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.
- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

BMC Monthly Contest 8, Problem 2
Evan O'Dorney, BMC Beginners I
Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into <https://www.gradescope.com/> before the deadline, Apr 26, 2023 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.
- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.
- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit <https://mathcircle.berkeley.edu/monthly-contest/contest-rules>.

Enjoy working on these problems and good luck!

Problems for Contest 8

1. A random integer is chosen from 0 to 500, inclusive. Without listing all of such integers, what is the probability that the integer is divisible by 7 or 11?
2. Given an even number of people, show that it is possible to arrange them in a circle so that no matter how the circle is cut in half, if the two halves of people stand facing each other, each person in one half is at least as tall as the person they face.
3. Find a prime that divides all numbers of the form $7^{2n+3} + 6^n$ for nonnegative integer n , and show it is unique.
4. Two permutations f and g of the integers from 1 to 15 are chosen. Given that $f(i) \neq g(i)$ for all i from 1 to 15, what is the maximum number of distinct integers that can be represented as the average of $f(i)$ and $g(i)$ for some i ?
5. Find the number of ways to connect 2023 islands numbered $1, 2, \dots, 2023$ with bridges such that there are no cycles, it is possible to get between any two islands, and along any path away from island 1 that doesn't backtrack, the island numbers increase.
6. Given a point P and two different lines ℓ and m which don't pass through P and are not parallel or at 45° or 90° angles to each other, how many squares can be drawn with one vertex at P and one vertex on each of ℓ and m ?
7. Aerith has finitely many traps. For the rest of eternity, each day she will either use a trap or save her traps for later. If Bob goes outside when there is a trap, he loses, but if he doesn't, he will learn the next day that Aerith tried to trap him. Show that regardless of Aerith's strategy, Bob can pick a day to go outside with a less than 1% chance of losing.