Berkeley Math Circle: Monthly Contest 4

Due January 31, 2024

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 4, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, January 31, 2024 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 4

1. Let S be the region in the Cartesian plane containing exactly the points satisfying

$$\begin{aligned} y &\geq 5x - 20, \\ y &\geq 0, \\ y &\geq 2 - x, \\ y &\leq 2x + 2, \\ 3y - 20 &\leq -x. \end{aligned}$$

   Let $f(x, y) = 2x - 3y$. Find the minimum and maximum of f as it ranges across S.

2. Let a be a three-digit number with middle digit 0. Another three-digit number b is formed by reversing the digits of a. Prove that $a + b$ cannot be a perfect square.

3. Let $ABCD$ be a cyclic quadrilateral with ω and O being its circumcircle and circumcenter, respectively. Suppose that $AC \perp BD$. Let P be the intersection of the diagonals AC and BD Given that $AP = DP = 1$ and $BP = CP = 2$, compute the area bounded by BP, CP, and ω in terms of $\angle BOC$, represented in radians.

4. What is the period of the $\frac{2023_6}{114202_6}$ when written as a repeating decimal, when expressed in base 6? Here, the subscript of 6 represents base-6 notation.

5. Aerith and Bob play a game on an infinite lattice. On each of their turns, they may write 14 or 16 on a square of the lattice, 8 or 10 on a vertex, or an integer between 1 and 5, inclusive, on an edge. Bob wins if there are four numbered edges each adjacent to a numbered vertex whose sum is less than or equal to the number on the vertex, and where no more than two edges share a number. Aerith wins if there are four numbered edges each adjacent to a numbered square whose sum is greater than or equal to the number on the square, and where no more than two edges share a number. Show that Aerith can always stop Bob from winning if she goes first.

6. Let p be an odd prime, and define the polynomial $f(x) = x^{p+1} + (1 - p)x^p - p$.
   (a) Prove that $x + 1$ divides $f(x)$.
   (b) Let $g(x) = \frac{f(x)}{x+1}$. Prove that $g(x)$ is irreducible.

7. A *toroidal helix* is a curve $c(t)$ on a torus so that each angle function is a linear function of t. Note that a torus, thought of as a circle revolved around a line, has two angle functions, namely the angle of revolution and the angle on the circle.
   (a) How many ways can one place a rectangular grid with 130 squares on a torus? The rectangular grid is generated by dividing the torus into 130 regions, defined by two families of non-intersecting toroidal helices that always meet at 90°.
   (b) What about a hexagonal grid with 105 hexagons on a torus? The hexagonal grid is generated similarly, using 3 families of non-intersecting toroidal helices that always meet at 60°.

   Two grids are considered the same if the induced graph structures are equivalent, so that there is a one-to-one correspondence between regions that preserves adjacency.

   Hint: read about universal covering manifolds, flat tori, and Dedekind's second proof (1894) of Fermat's Theorem on sums of squares.