Berkeley Math Circle: Monthly Contest 2

Due October 25, 2023

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 2, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, October 25, 2023 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 2

1. Suppose that m is a two-digit positive integer. Let us form a new six-digit integer n by appending the two digits of m in the end of m twice. For example, if m is equal to 73, then n is 737373. Find four pairwise distinct prime numbers p, q, r, and s, all independent of the choice of m such that $pqrs \mid n$.

2. Let $ABCD$ be a quadrilateral with $AC \perp BD$, and define E to be the intersection of diagonals AC and BD. Let A', B', C', and D' be the projections of E onto sides AB, BC, CD, and DA, respectively. Prove that $A'B'C'D'$ is cyclic.

3. Aerith and Bob take turns naming prime numbers. Aerith starts with the number 2. Each turn, a player must add a positive integer to the current number to get another prime number, adding no more than twice what is necessary to do so. In other words, if the current number is n and the next prime number is p, the player may not add more than $2(p-n)$.

   Both players want to be the one who says 4567. Who has a winning strategy?

4. Let $p(x) = x^3 - ax^2 + bx - c$ be a polynomial for some positive real numbers a, b, and c. Prove that if $p(x)$ has three positive roots, counting multiplicity,
$$a^2b^2 \geq 2b^3 + 27c^2.$$

5. The center of a circle lies on the bisector of quadrant I of the coordinate system. The distance between the center of the circle and the origin of the coordinate system is d, and the radius of the circle is r, where both r and d are independent random variables uniformly distributed in $[0,1]$. Find the probability that part of the circle lies in quadrants II and IV, but not in quadrant III.

6. For which real number a_0 does the recursive sequence defined by $a_n = 1 + na_{n-1}$ converge to zero?

7. Consider an assignment of integers to each edge of a regular dodecahedron, so that the following criteria are met.

   a) At each vertex, the sum of the numbers on the edges leading to that vertex is a multiple of 3.

   b) At each vertex, the product of two of the numbers on the edges leading to that vertex leaves a remainder of 2 when divided by 3.

   Prove that there are exactly 3 pairs of opposite edges whose numbers leave the same remainder when divided by 3, and show that each of these 3 remainders are distinct.

   The diagram shown below is a planar representation of a solid dodecahedron.