Berkeley Math Circle: Monthly Contest 1

Due September 27, 2023

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 1, Problem 2
Evan O'Dorney, BMC Beginners I
Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, September 27, 2023 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 1

1. If integers a, b, c, d are such that $a + b = c + d$, prove that $abcd + 1$ is not a multiple of three.

2. Let A, B, C, D, E be five points in the plane satisfying $AB = BC = CD = DE = EA$ and $AC = CE = EB = BD = DA$. Show that $ABCDE$ is a regular pentagon.

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(x) + f(y) - f(x + y) = f(xy + 1)$ for all integers x and y.

4. Jessica rolls a fair die until she gets three 6s in a row, at which point she stops. What is the expected number of times that Jessica will roll the die?

5. Evan Corporation LLC has 956 employees, 24 departments, and a total budget of \$215100. If V. Enhance, the company's CEO, sends m employees and n dollars to a certain department, then that department will generate a total revenue of $100\sqrt{mn}$ dollars annually. What is the maximum possible total revenue that Evan Corporation LLC can generate annually? Note that Mr. Enhance does not count as an employee.

6. A gambler starts with an initial amount of \$3 and makes a series of bets. The gambler earns \$1 upon winning a bet and loses \$1 otherwise. The gambler stops betting when he either reaches a total amount of \$6 or completely runs out of money. Letting p be the probability of the gambler losing an arbitrary individual bet, independent of all other bets, compute, with proof, the probability that gambler eventually goes broke.

7. A sequence of real numbers a_n satisfies $\min(a_m, a_n) = a_{\gcd(m,n)}$ for all positive integers m, n. Must we have $a_1 + \cdots + a_{26} \leq a_{2023} + \cdots + a_{2048}$?