Berkeley Math Circle: Monthly Contest 8

Due May 4, 2022

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 8, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, May 4, 2022 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 8

1. If $a < b < c < d$ are distinct positive integers such that $a + b + c + d$ is a square, what is the minimum value of $c + d$?

2. An unfair coin comes up heads with probability $\frac{4}{7}$ and tails with probability $\frac{3}{7}$. Aerith and Bob take turns flipping the coin until one of them flips tails, with Aerith going first. What is the probability that Aerith wins the game?

3. Unit segments AB and CD are given such that $AB \nparallel CD$. If M is the midpoint of AC, N the midpoint of BD, and X the intersection of lines AB and CD, given $M \neq N$ and $A \neq X \neq C$, show that the angle bisector of $\angle AXC$ is either parallel or perpendicular to MN.

4. Aerith picks five numbers and for every three of them, takes their product, producing ten products. She tells Bob that the nine smallest positive divisors of sixty are among her products. Can Bob figure out the last product?

5. Aerith writes a positive integer in each cell of a 2021×2021 grid. Every second, Bob will pick a cell with value at least five if one exists, decrease its value by four, and increment each of the cell's neighbors by 1. Must this process always stop?

6. The number 0 is written on a blackboard. Every minute, Aerith simultaneously replaces every 0 with a 1 and every 1 with a 10. For example, if the current number were 1100, in one minute it would be 101011. She eventually gets tired and leaves, leaving some number N written on the board. If $9|N$, show that $99|N$.

7. A certain rectangle can be tiled with a combination of vertical $b \times 1$ tiles and horizontal $1 \times a$ tiles. Show that the rectangle can be tiled with just one of the two types of tiles.