

Berkeley Math Circle: Monthly Contest 6

Due March 16, 2022

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.
- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

BMC Monthly Contest 6, Problem 2
Evan O'Dorney, BMC Beginners I
Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into <https://www.gradescope.com/> before the deadline, March 16, 2022 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.
- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.
- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit <https://mathcircle.berkeley.edu/monthly-contest/contest-rules>.

Enjoy working on these problems and good luck!

Problems for Contest 6

1. On Predictable Island, everyone is either a liar (who always lies) or a truth-teller (who always tells the truth). You encounter Aerith who says, "Bob and I are both liars." What are they actually?
2. Bob has five airplane tickets with prices \$100, \$120, \$140, \$160, and \$180. Bob gives an offer to Aerith: she can distribute his tickets among two bags, after which, without looking inside, Bob will randomly choose a bag and a ticket from it for Aerith to keep. What strategy should Aerith use to maximize the expected value of her ticket?
3. In $\triangle ABC$, let O be the circumcenter, D the foot of the altitude from A to BC , and E the foot of the altitude from B to AC . Show that $DE \perp CO$.
4. Find the remainder when $10^{2^0} + 10^{2^1} + \dots + 10^{2^{2021}}$ is divided by $\underbrace{44 \cdots 44}_{44}$.
5. Aerith has an unlimited number of coins of values 5, $m - 1$, and m , and 71 is the greatest value that cannot be made with a combination of these coins. Find all possible values of m .
6. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{R}$ from rational to real numbers such that for all rational p, q ,
$$f(p + 2q) - f(p - 2q) = 2(f(p + q) - f(p - q)).$$
7. In the game Pogémon, Pogé Balls contain Pichagus with a 50% chance and otherwise contain nothing. The new Pogémon universe started with one Pichagu and has existed for five hours. Every hour, every Pichagu at the time opened a new Pogé Ball.
 - a) What is the expected number of Pichagus created in the last hour?
 - b) Every two currently existing Pichagus will now battle each other. What is the expected number of battles?