Berkeley Math Circle: Monthly Contest 5

Due February 16, 2022

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.
- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 5, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, February 16, 2022 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.
- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.
- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 5

1. In a trapezoid, the midsegment has length 17 and the distance between the midpoints of the diagonals is 7. Find the lengths of the bases.

2. Aerith has 5 coins, all with heads facing up. She wants to flip them so that they all have tails facing up.

   a) If she must flip exactly three coins at a time (from heads to tails or vice versa), is this possible?

   b) What if she must flip exactly two coins at a time?

   For each part, either show a way it can be done or prove that it is impossible.

3. Show that $a^{1729} \equiv a \pmod{1729}$ for all positive integers a.

4. How many solutions does $26 = twelve + eleven + two + one$ have over the positive integers? (Each letter is a variable, and letters in the same word are multiplied.)

5. Aerith writes 100 positive numbers on a blackboard. Every minute, Bob either replaces one number x with $\frac{1}{x}$, or replaces two numbers x, y with $\frac{xy+1}{x+y}$. Given that after 2021 minutes, there is only one number left, show that this number only depends on Aerith's initial numbers.

6. Show that the sum $AP^4 + BP^4 + CP^4$ does not depend on P, where P is a point on the circumcircle of equilateral triangle $\triangle ABC$.

7. "Very Frustrating Game" has six levels. When a level is attempted, the player goes to the next level if they succeed, but back to the previous level if they fail (or if they are on level 1 they restart).

   a) Aerith has a $\frac{1}{2}$ success rate on all levels. How many level attempts on average would it take her to complete the game?

   b) Bob has a $\frac{1}{3}$ success rate on all levels. How many level attempts on average would it take him to complete the game?