Berkeley Math Circle: Monthly Contest 4

Due Jan 18, 2023

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 4, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, Jan 18, 2023 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 4

1. Find the smallest prime number with digit sum n or show that none exists, if n is:
   (a) 13.
   (b) 14.
   (c) 15.

2. A solid is formed by gluing a prism to a pyramid so that the base of the pyramid lines up completely with one base of the prism. Is it possible for the solid to have exactly 2023:
   (a) faces?
   (b) vertices?
   (c) edges?

3. Marquis owns a factory where he produces bottled orange juice. He is preparing a shipment of 1,000,000 bottles of orange juice to send to his stores when he learns that a mistake in the factory caused one of the bottles to be poisoned! No one knows which bottle it might be, but Marquis has 20 poison testing devices to test the bottles. Each tester can only be used once, and he can administer as much orange juice as he wants to each of them. How can Marquis use the testers to determine which bottle is poisoned?

4. A polynomial of degree 2022 satisfies $P(n) = 2022^n$ for $n = 0, 1, \ldots, 2022$. What is $P(2023)$?

5. Aerith and Bob are playing a game on the edges of an infinite square grid. They alternate turns coloring one uncolored edge amber or bronze, respectively. Aerith wins if she forms a cycle, and Bob wins if he can prevent this indefinitely. Does Aerith have a winning strategy?

6. Show that no primes $2 < p < q < r$ exist so that $\frac{r-q}{p}$ and $\frac{r-p}{q}$ are both perfect squares.

7. Andy the ant is crawling along the edges of a 4×4 grid. At each step, he chooses one of the horizontally or vertically adjacent vertices to crawl to with equal probability. For each starting point, what is the probability he reaches the bottom left corner A before the top right corner B?