Berkeley Math Circle: Monthly Contest 1

Due September 28, 2022

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 1, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, September 28, 2022 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 1

1. A straight ladder starts upright against a vertical wall, and slides down until it is horizontal such that the top is always along the wall and the bottom on the floor. What shape does its midpoint trace out?

2. Sofiya and Marquis play a game by taking turns. They form a circle with 2023 other people, and on each turn Sofiya or Marquis can remove one of their neighbors to the left or to the right from the circle. The person who removes the other player wins. If Sofiya starts, who has the winning strategy?

3. Let a and b be integers. Show that 29 divides $3a+2b$ if and only if it divides $11a+17b$.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ from the integers to the integers satisfying

$$f(m + f(n)) - f(m) = n$$

for all integers $m, n \in \mathbb{Z}$.

5. Find all positive integers n such that $n^4 - 27n^2 + 121$ is a prime positive integer.

6. If $f(n, k)$ is the number of ways to divide the set $\{1, 2, \ldots, n\}$ into k nonempty subsets and m is a positive integer, find a formula for $\sum_{k=1}^{n} f(n, k)m(m-1)(m-2)\cdots(m-k+1)$.

7. For points P, Q, R, let $E_{P,Q}(R)$ denote the ellipse with foci P and Q through R.

   Points T, A, B, C are points on a line in that order. A ray from A intersects $E_{A,B}(T)$ at Q and $E_{A,C}(T)$ at P. Show that $E_{B,P}(A)$ intersects $E_{C,Q}(A)$ at two points.