Berkeley Math Circle: Monthly Contest 8

Due May 5, 2021

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 8 is due on May 5, 2021.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 8, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 8

1. If $P_1P_2\ldots P_{100}$ is a regular 100-gon, what is the measure of the angle $\angle P_{20}P_2P_1$ in degrees?

2. For which positive integers n is n^4+4 equal to a prime number?

3. Find all real numbers x for which $\tan(x/2)$ is defined and greater than $\sin(x)$.

4. Aerith and Bob take turns picking a nonnegative integer, each time changing exactly one digit from the other's last number. The first person to pick a number that (s)he picked before loses. If Aerith goes first, and both play optimally, who wins?

   (Note: There are no leading zeroes, except in the number 0 itself. For instance, if one person picks 2020, the other could respond by picking $0020 = 20$, however the reverse does not hold.)

5. A set S of irrational real numbers has the property that among any subset of five numbers in S, one can find two with irrational sum. How large can $|S|$ be?

6. Show that if n is a positive integer for which $2 + 2\sqrt{1 + 12n^2}$ is an integer, then it is a perfect square.

7. Let ABC be an acute scalene triangle with centroid G. The rays BG and CG meet the circumcircle of ABC again at points P and Q. Let D denote the foot of the altitude from A to BC. Suppose ray GD meets the circumcircle of ABC again at E. Show that the circumcircle of triangle ADE lies on line PQ.