Berkeley Math Circle: Monthly Contest 7

Due April 14, 2021

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 7 is due on April 14, 2021.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 7, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 7

1. Given a quadrilateral $ABCD$, show that the midpoints of its four edges form the vertices of a parallelogram.

2. Let x, y, z be nonzero real numbers such that the equations

$$x + \frac{1}{y} = y + \frac{1}{x}$$
$$y + \frac{1}{z} = z + \frac{1}{y}$$
$$z + \frac{1}{x} = x + \frac{1}{z}$$

   all hold. Show that two of the three variables must be equal.

3. There are m friends with n cupcakes each weighing 1 ounce. They wish to split the cupcakes equally by dividing each cupcake into some number of parts, and allocating some parts to each person.

a) Assume $m = 3$ and $n = 5$. Show that they may divide the cupcakes with all pieces being larger than $\frac{1}{3}$ ounces.

b) Assume $m = 5$ and $n = 3$. Show that they may divide the cupcakes with all pieces being larger than $\frac{1}{5}$ ounces.

4. Eight friends, Aerith, Bob, Chebyshev, Descartes, Euler, Fermat, Gauss, and Hilbert, bought tickets for adjacent seats at the opera. However when they arrived they mixed up their seats:
   - Bob sat in his assigned seat,
   - Chebyshev sat two seats to the right of Gauss' assigned seat,
   - Descartes sat one seat to the left of Fermat's assigned seat,
   - Euler sat four seats to the left of Hilbert's assigned seat,
   - Fermat sat five seats to the right of Descartes' assigned seat,
   - Gauss sat one to the right of Euler's assigned seat,
   - Hilbert sat three seats to the left of Aerith's assigned seat.

   In whose seat did Aerith sit?

5. Let n be a positive integer which also divides $2^n - 1$. Show that $n = 1$.

6. Let ABC be an acute triangle with circumcenter O, incenter I, orthocenter H. If $OI = HI$, what are the possible values of the angles of triangle ABC?

7. Prove that if positive real numbers x, y, z have sum 1, then

$$\frac{x}{x+yz} + \frac{y}{y+zx} + \frac{z}{z+xy} \leq \frac{2}{1-3xyz}.$$