Berkeley Math Circle: Monthly Contest 7
Due April 14, 2021

Instructions

• This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Contest 7 is due on April 14, 2021.

• Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

   BMC Monthly Contest 7, Problem 2
   Evan O’Dorney
   Grade 3, BMC Beginner
   from Springfield Middle School, Springfield

Submit different problems on different pages as they are graded separately.

• Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

• Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at http://mathcircle.berkeley.edu for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 7

1. Given a quadrilateral $ABCD$, show that the midpoints of its four edges form the vertices of a parallelogram.

2. Let $x, y, z$ be nonzero real numbers such that the equations

   \[ \frac{1}{y} + x = y + \frac{1}{x} \]
   \[ y + \frac{1}{z} = z + \frac{1}{y} \]
   \[ z + \frac{1}{x} = x + \frac{1}{z} \]

all hold. Show that two of the three variables must be equal.

3. There are $m$ friends with $n$ cupcakes each weighing 1 ounce. They wish to split the cupcakes equally by dividing each cupcake into some number of parts, and allocating some parts to each person.
a) Assume $m = 3$ and $n = 5$. Show that they may divide the cupcakes with all pieces being larger than $\frac{1}{3}$ ounces.

b) Assume $m = 5$ and $n = 3$. Show that they may divide the cupcakes with all pieces being larger than $\frac{1}{5}$ ounces.

4. Eight friends, Aerith, Bob, Chebyshev, Descartes, Euler, Fermat, Gauss, and Hilbert, bought tickets for adjacent seats at the opera. However when they arrived they mixed up their seats:
   - Bob sat in his assigned seat,
   - Chebyshev sat two seats to the right of Gauss’ assigned seat,
   - Descartes sat one seat to the left of Fermat’s assigned seat,
   - Euler sat four seats to the left of Hilbert’s assigned seat,
   - Fermat sat five seats to the right of Descartes’ assigned seat,
   - Gauss sat one to the right of Euler’s assigned seat,
   - Hilbert sat three seats to the left of Aerith’s assigned seat.
   In whose seat did Aerith sit?

5. Let $n$ be a positive integer which also divides $2^n - 1$. Show that $n = 1$.

6. Let $ABC$ be an acute triangle with circumcenter $O$, incenter $I$, orthocenter $H$. If $OI = HI$, what are the possible values of the angles of triangle $ABC$?

7. Prove that if positive real numbers $x, y, z$ have sum 1, then

$$\frac{x}{x + yz} + \frac{y}{y + zx} + \frac{z}{z + xy} \leq \frac{2}{1 - 3xyz}.$$