Berkeley Math Circle: Monthly Contest 6

Due March 17, 2021

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 6 is due on March 17, 2021.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 6, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 6

1. Find all nonzero real numbers x such that
$$x^2 + \frac{36}{x^2} = 13.$$
2. Prove or disprove the following assertion: given 3 noncollinear points in the plane, the disk with the smallest radius which contains all three points is the circle passing through all three points.
3. For any positive integer $n \geq 3$, show that you can write 1 as a sum of n fractions with numerator 1 and different denominators.
4. Aerith and Bob are playing "Not Quite Tic-Tac-Toe", in which an X is written on a line, and each player takes tuns adding either an X or an O (their choice) to the end of the line. Aerith goes first, and the goal is to avoid a sequence of three evenly spaced X's or O's; the first person to do so loses.

For instance, if the letters on the line are currently XOXXOX, Bob is forced to write down O in order to avoid $\underline{\text{X}}\text{OX}\underline{\text{X}}\text{OX}\underline{\text{X}}$. Aerith would then lose, as both $\text{XOX}\underline{\text{X}}\text{O}\underline{\text{X}}\text{O}\underline{\text{X}}$ and $\text{X}\underline{\text{O}}\text{XX}\underline{\text{O}}\text{XO}\underline{\text{O}}$ are losing positions.

Assuming that both of them play optimally after Aerith's first move, who wins if she starts by putting down a second X on the line next to the initial one? What if she starts with an O?

5. Does there exist a function f from the positive integers to itself, such that for any positive integers a and b, we have $\gcd(a, b) = 1$ if and only if $\gcd(f(a), f(b)) > 1$ holds?

6. Equilateral triangles $\triangle ABD$, $\triangle ACE$, and $\triangle BCF$ are drawn outside $\triangle ABC$ on each of its sides, with centers G, H, and I, respectively.
   a) Prove that $\triangle GHI$ is equilateral.
   b) Let G', H', and I' be the reflections of G, H, and I across AB, BC, and CA, respectively. Prove that $\triangle GHI$ and $\triangle G'H'I'$ have the same circumcenter.

7. Do there exist positive integers $a_1, \ldots, a_{100}$ such that for each $k = 1, \ldots, 100$, the number $a_1 + \cdots + a_k$ has exactly a_k divisors?