Berkeley Math Circle: Monthly Contest 5

Due February 17, 2021

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 5 is due on February 17, 2021.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 5, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 5

1. A polygon is *regular* if all its sides and angles are the same. Find the measure of each angle in a regular dodecagon (12-sided polygon).

2. Prove that if x is an positive real number such that $x + x^{-1}$ is an integer, then $x^3 + x^{-3}$ is an integer as well.

3. Show that for any positive integers a, b, and c,

$$abc \operatorname{lcm}(a, b, c) \geq \operatorname{lcm}(b, c) \operatorname{lcm}(c, a) \operatorname{lcm}(a, b),$$

   where lcm denotes the least common multiple.

4. Aerith and Bob play rounds of pool. At some point Bob had won more rounds than Aerith, but now Aerith has won 85% of their rounds. Show that at some point, Aerith had won exactly 75% of their rounds.

5. In a regular n-gon, all the diagonals are drawn, forming smaller regular n-gons inside. If the outer regular n-gon has side length 1, show that the kth largest regular n-gon formed has side length

$$\frac{\cos(k\pi/n)}{\cos(\pi/n)}$$

(where the original regular n-gon is the 1st largest).

6. Can $x^{2020} - 8$ be written as the product of two nonconstant polynomials with integer coefficients?

7. Prove that there exists infinitely many integers n such that $n^4 + 2020$ has a prime divisor larger than $2n$.