Berkeley Math Circle: Monthly Contest 4

Due January 19, 2022

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 4, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, January 19, 2022 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 4

1. Aerith bakes some cookes. On the first day, she gives away 1 cookie and then 1/8 of the remaining cookies; on the second day, she gives away 2 cookies and then 1/8 of the remaining cookies, and so on. On the 7th day, she gives away 7 cookies and then there are none left. How many cookies did she bake?

2. Points A, B, C, M are such that B, M, C lie on a line, $AB = AC = 353$, and $BM = MC$. Point I is on segment AM such that the circle centered at I through M has radius 106 and is tangent to AB and AC. Given that there are two possible areas of right triangle AMB, find the larger one.

3. Let A and B be diagonally opposite vertices of a cube. An ant is crawling on a cube starting from A, and each second it moves at random to one of the three vertices adjacent to its current one. Find the expected number of steps for the ant to get to vertex B.

4. Let $S \subset \mathbb{N}$ be a set of positive integers whose product is 2021 times its sum.
   a) Given that S has five (or more) elements, what its minimum possible sum?
   b) Given that S has exactly four elements, what its minimum possible sum?
   c) Given that S has exactly three elements, what its minimum possible sum?

5. Determine all complex numbers w such that
$$10|w|^2 = 2|w+2|^2 + |w^2+1|^2 + 20.$$

6. Squares $MATH$, $GREW$, $UNDO$ are such that MUG, RAN, TED are all lines. Show that WHO is a line as well.

7. In a country with n cities, there is a recurrent one-way flight between every pair of cities. Every flight has a constant price in the range $\$100, \$120, \$140, \$160, \$180$. A $\$N$ flight ticket gives unlimited access to flights which cost $\$N$, and tickets can be traded for tickets of lower prices. For example, with a \$160 ticket, Bob could take a \$160 flight, trade his ticket for a \$120 ticket, then take a \$120 flight.

   Aerith loves flying and wonders how many successive flights she can take with one ticket. What is the minimum n needed to guarantee that she can take 4 such flights in a row?