Berkeley Math Circle: Monthly Contest 3

Due December 1, 2021

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 3, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, December 1, 2021 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 3

1. Find four different ways to write 24 using the numbers 4, 5, 7, 8 (each exactly once). You can use addition, subtraction, multiplication, division, and parentheses. Reordering numbers that are added or multiplied together does not count as different.

2. Let x, y, z be positive real numbers with $x + 2y + 3z = 1$. Find the maximum value of $\min(2xy, 3xz, 6yz)$.

3. Suppose chameleons are either red, green, or blue, and whenever two of them of different colors meet, they both change to the third color. Otherwise, they don't change colors. Suppose you start with 12 red chameleons, 13 green chameleons, and 14 blue chameleons. Is it possible that at some point all the chameleons will be red?

4. a) Can Aerith put 48 points in the interior of a 6×6 square such that no two are at most one unit apart?

   b) Can Bob put 68 points in the interior of a 7×7 square such that no two are at most one unit apart?

5. What is the smallest nonnegative integer n for which $\frac{(20n)!}{(n!)^{21}}$ is not an integer?

6. Bob is thinking of a positive integer under 100. Aerith wants to figure out his number, and can ask him seven yes or no questions about it. However, if Bob says no four times, he will feel too sad to answer any more questions. Show that Aerith can always figure out his number.

7. Suppose x, y, and z are nonzero real numbers. Let

$$\begin{aligned} a &= 50x^2 + 9y^2 + 5z^2 + 6xy + 30xz + 6yz, \\ b &= 25x^2 + 41y^2 + 5z^2 + 40xy + 20xz + 26yz, \\ c &= 35x^2 + 15y^2 + 5z^2 + 33xy + 25xz + 16yz. \end{aligned}$$

   Given that $c^2 = ab$, find all possible values of $\frac{x}{y}$.