Berkeley Math Circle: Monthly Contest 2

Due October 27, 2021

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 2, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, October 27, 2021 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 2

1. Find the units digit of 17^{2021}.

2. Let $ABCD$ be a convex quadrilateral. Let I_A be the incenter of BCD and define I_B, I_C, I_D similarly. Show that $AC \perp I_B I_D$ if and only if $I_A I_C \perp BD$.

3. Cheryl chooses a word in this problem and tells its first letter to Aerith and its last letter to Bob.[1][2] The following conversation ensues over a series of emails:

   Aerith: "I don't know her word, do you?"
   Bob: "No, in fact, I don't know if we can ever figure out what her word is without having more information."
   Aerith: "Then I do know what it is!"
   Bob: "Now I also do."

   What is Cheryl's word?

4. Find all polynomials f that satisfy the equation $\frac{f(9x)}{f(3x)} = \frac{243x-729}{x-81}$ for infinitely many values of x.

5. Without a calculator, find a factor $85^9 - 21^9 + 6^9$ that is between 2000 and 3000.

6. For which $n \geq 3$ can an $n \times n$ board with all four corners removed be tiled with "L" and "J" tetris pieces?

7. Aerith and Bob are playing tag at Lake Round, a perfectly circular lake. Aerith tags Bob right next to the lake and dives in. Aerith can swim at a speed of 2mph, while Bob can't swim but runs at a speed of 9mph. Can Aerith leave the lake without getting tagged?[3]

[1] Aerith and Bob are both aware of this setup.
[2] Footnotes are not considered to be part of the problem statement.
[3] Assume that they always know each other's locations and can instantaneously accelerate.