Berkeley Math Circle: Monthly Contest 1

Due September 29, 2021

Instructions (Read carefully)

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (intended for grades 9–12). Younger students are also eligible for and will automatically be entered into the advanced contest if they receive a top score on the last 5 problems.

- Each problem is worth 7 points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.

- You may type up your solutions or write them by hand. Use separate page(s) for each problem, as they are graded separately. Begin each solution with the contest number, problem number, your name, BMC group, grade level, and school. An example header:

  BMC Monthly Contest 1, Problem 2
  Evan O'Dorney, BMC Beginners I
  Grade 3, Springfield Middle School, Springfield

- Every BMC student should have received an email invitation to join this year's BMC Monthly Contest course on Gradescope. Submit your solutions by logging into `https://www.gradescope.com/` before the deadline, September 29, 2021 at 11:00PM. There is a one-hour grace period to resolve any last-minute technical issues, but if you have not yet created your Gradescope account you should do so well ahead of this deadline to sort out any account or access issues.

- If you typed your solutions or if you have access to a scanner, submitting a single PDF file is preferred; otherwise you can take a picture of each page and submit these individually. Be sure that your phrasing is clear and that your writing is legible and in focus - no credit can be given for your hard work if it cannot be understood by the graders. As part of the submission process, you are asked to assign problem numbers to each page of your submission. *This step is important*, as the grader will not otherwise see your submission when working on a particular problem.

- Three winners are awarded from each of the Beginner and Advanced contests. Winners from last month's contest automatically receive a 7-point winner's handicap this time around. Should they continue to win despite this handicap they will receive a 14-point handicap at the next contest, and so on. This rule is to give more participants a chance to win and ultimately encourage broader participation.

- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. For the full contest rules, please visit `https://mathcircle.berkeley.edu/monthly-contest/contest-rules`.

Enjoy working on these problems and good luck!

Problems for Contest 1

1. Order the numbers $2^{300}, 10^{100}$, and 3^{200} from least to greatest, and prove that your ordering is correct.

2. Determine the number of convex polygons all of whose sides are the square roots of positive integers which can be inscribed in a unit circle. Polygons that are rotations or reflections of each other are considered the same.

3. Aerith and Bob play a game where they start with a row of 20 squares, and take turns crossing out one of the squares. The game ends when there are two squares left. Aerith wins if the two remaining squares are next to each other, and Bob wins if they are not next to each other.

   a) If Aerith goes first, who has a winning strategy?

   b) What if Bob goes first?

4. For positive integers a, b, c, x, y, z such that $axy = byz = czx$, can $a+b+c+x+y+z$ be prime?

5. Aerith rolls a fair die until she gets a roll that is greater than or equal to her previous roll. Find the expected number of times she will roll the die before stopping.

6. Find all positive integers N, n such that N^2 is 1 away from $n(N + n)$.

7. In $\triangle ABC$, suppose the incircle has center I and is tangent to BC at D, and the A-excircle has center I_a and is tangent to BC at D'. Show that ID' and I_aD intersect on the altitude from A to BC.