Berkeley Math Circle: Monthly Contest 8

Due April 29, 2020

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 8 is due on April 29, 2020.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 8, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 8

1. Suppose you have an unlimited number pennies, nickels, dimes, and quarters. Determine the number of ways to make 30 cents using these coins.

2. In a certain two-player game, you start with a rectangular $m \times n$ grid of squares. On each turn, a player either makes a horizontal cut and takes away the portion of the rectangle above the cut, or makes a vertical cut and takes away the portion to the right. Whichever player takes the last square (in the bottom left corner) loses. If both players play perfectly, determine for which values of m and n the first player will win.

3. A circle is inscribed in $\triangle ABC$ that touches side BC at D, side AC at E, and side AB at F. Show that $\triangle DEF$ must be acute.

4. Prove that every positive real number y satisfies

$$2y \geq 3 - \frac{1}{y^2}.$$

When does equality occur?

5. It's a week before Thanksgiving, and a family is trying to find their turkey. There are 5 boxes in a row, and the turkey is hiding in one of the 5 boxes.

   Every day, the family is allowed to check one box to try to find the turkey, and every night, the turkey moves to a box right next to the box it was in. For example, from box 3 it could move to box 2 or 4, and from box 5 it must move to box 4. Determine a strategy for the family to catch the turkey before Thanksgiving.

6. A piece of origami paper is green on one side and white on the other. The paper has been folded along four or more straight line segments, all meeting at the same vertex, such that the folded model lies flat. A fold is a **mountain fold** if the white sides of the paper are touching and the green is on the outside, and a **valley fold** if the green sides are touching and the white is on the outside.

   a) The line segments meeting at the vertex form a number of angles, one between each adjacent pair of segments. Suppose we color half the angles white and half the angles black such that no two angles next to each other are the same color. Show that the sum of the black angles is 180°.

   b) Show that the number of mountain folds and the number of valley folds meeting at the vertex differ by exactly 2.

7. In the pattern shown below, row 1 (the bottom row) consists of two 1's, and row n is formed by taking row $n-1$ and inserting between each adjacent pair of numbers a and b their sum $a+b$:

1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
1		4		3		5		2		5		3		4		1
1				3				2				3				1
1								2								1
1																1

   In row 2019 of the pattern, how many copies of 2019 will there be?