Berkeley Math Circle: Monthly Contest 7

Due April 8, 2020

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 7 is due on April 8, 2020.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 7, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 7

1. The numbers 1 to 10 are written on a blackboard in a row. Determine whether it is possible to put nine $+$ and $-$ signs between them (one between each pair of adjacent numbers) to make the resulting sum 0.

2. In a certain kingdom, the only coin values are 3 and 5. Determine all possible amounts of money you can have using only these coins.

3. How many ways are there to list the numbers 1 to 10 in some order such that every number is either greater or smaller than all the numbers before it?

4. A large integer is divisible by all the integers between 1 and 30 inclusive, except for two consecutive integers. Determine those two consecutive integers.

5. Prove that every positive real number x satisfies

$$\sqrt{x^2 - x + \frac{1}{2}} \geq \frac{1}{x + \frac{1}{x}}.$$

6. On a certain island, there are knights, who always tell the truth, knaves, who always lie, and spies, who could do either. Suppose you meet three people, and you know one is a knight, one is a knave, and one is a spy, but you don't know which is which. Find a method to ask three yes/no questions, each to one of the three people, so you can determine for sure which is which. You may question the same person multiple times, and your questions can depend on answers to previous questions.

7. Given ten points in the plane, show that it is always possible to cover all of them with non-overlapping unit circles.