Berkeley Math Circle: Monthly Contest 6

Due March 11, 2020

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 6 is due on March 11, 2020.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 6, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 6

1. At a party with 100 people, everyone is either a knight, who always tells the truth, or a knave, who always lies. Each person says they shook hands with a different number of knights at the party, from 0 to 99. Each pair of people shook hands at most once, and everyone knows whether each other person is a knight or knave. Determine how many knights were at the party.

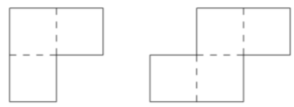
2. A castle has infinitely many rooms labeled $1, 2, 3, \ldots$, which are divided into several halls. Suppose room n is on the same hall as rooms $3n + 1$ and $n + 10$ for every n. Determine the maximum possible number of different halls in the castle.

3. Two points A and C are marked on a circle; assume the tangents to the circle at A and C meet at P. Let B be another point on the circle, ans suppose PB meets the circle again at D. Show that $AB \cdot CD = BC \cdot DA$.

4. Suppose you have three children and 40 pieces of candy. How many ways are there to distribute the candy such that each child gets more than one but fewer than 20 pieces?

5. Another castle has infinitely many rooms labeled $1, 2, 3, \ldots,$ divided into several halls. Suppose room n is on the same hall as rooms $2n + 1$ and $8n + 1$ for every n. Determine the maximum possible number of different halls in the castle.

6. A 99×99 rectangular region is to be tiled with tiles of the following shape:



   We allow rotations and reflections, but the tiles must remain parallel to the edges of the rectangle, and they must fit in the rectangle without overlap, covering each cell. Prove that this is possible and determine the minimum number of tiles which must be used.

7. Suppose there are 100 prisoners, each of whom is given a number between 1 and 100. There is also a room with 100 boxes, labeled 1 to 100, and 100 pieces of paper with the numbers 1 to 100 on them. Each piece of paper is randomly placed inside one of the 100 boxes.

   One at a time, each prisoner is allowed to enter the room and open at most 50 boxes to see the numbers inside. If every prisoner opens the box with his own number inside it, they will all be released. They are not allowed to communicate at all during this process, but they can come up with a strategy beforehand. Show that there is a strategy that gives them at least a 30% chance of winning.