Berkeley Math Circle: Monthly Contest 5

Due February 12, 2020

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 5 is due on February 12, 2020.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 5, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 5

1. A *palindrome* is a positive integer that reads the same forward and backward, like 2552 or 1991. Find a positive integer greater than 1 that divides all four-digit palindromes.

2. A sheet of graph paper has perpendicular grid lines spaced 1 unit apart. On the paper, you draw a polygon all of whose edges lie along the grid lines. Determine all possible perimeters for this polygon.

3. Suppose x, y, and z are real numbers that satisfy $x + y + z > 0$, $xy + yz + zx > 0$ and $xyz > 0$. Prove that x, y, and z must all be positive.

4. You are blindfolded and have a spinning table with four switches on it in front of you. The switches are always either up or down, and you don't know what configuration they start in. On each move, you can spin the table some unknown amount, then reach out and choose two switches (either next to each other or diagonally opposite

each other), feel whether they are up or down, and then choose to flip one, both, or neither of them. You win if you turn the switches either all up or all down. Determine whether it is always possible to win the game.

5. Find a formula for the sum of the squares of the numbers in the nth row of Pascal's triangle (i.e. the numbers $\binom{n}{0}$, $\binom{n}{1}$, $\ldots$, $\binom{n}{n}$).

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers $x \neq 0$,

$$f(x) + 2f\left(\frac{x-1}{x}\right) = 3x.$$

   (Note that the function must be defined for all $x \in \mathbb{R}$, including $x = 0$.)

7. An infinite castle has rooms labeled $1, 2, 3, \ldots$. If room n is on the same hall as rooms $2n+1$ and $3n+1$ for every n, what is the maximum possible number of different halls on the castle?