Berkeley Math Circle: Monthly Contest 4

Due January 22, 2020

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 4 is due on January 22, 2020.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 4, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 4

1. Prove that $\sqrt{n+1}+\sqrt{n}$ is irrational for every positive integer n.

2. Suppose a sequence s_1, s_2, $\dots$, of positive integers satisfies $s_{n+2} = s_{n+1} + s_n$ for all positive integers n (but not necessarily $s_1 = s_2 = 1$). Prove that there exists an integer r such that $s_n - r$ is not divisible by 8 for any integer n.

3. For positive real numbers a, b, c satisfying $ab + bc + ca = 1$, prove that
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq a^2+b^2+c^2+2.$$

4. Suppose we have a convex polygon in which all interior angles are integers when measured in degrees, and the interior angles at every two consecutive vertices differ by exactly 1°. If the greatest and least interior angles in the polygon are M° and m°, what is the maximum possible value of $M - m$?

5. Given a quadrilateral $ABCD$ extend AD and BC to meet at E and AB and DC to meet at F. Draw the circumcircles of triangle ABE, ADF, DCE, and BCF. Prove that all four of these circles pass through a single point.

6. Determine, with proof, whether or not there exist distinct positive integers a_1, a_2, $\ldots$, a_n such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 2019.$$

7. A simple graph G on 2020 vertices has its edges colored red and green. It turns out that any monochromatic cycle has even length. Given this information, what is the maximum number of edges G could have?