Berkeley Math Circle: Monthly Contest 3

Due December 2, 2020

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 3 is due on December 2, 2020.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 3, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

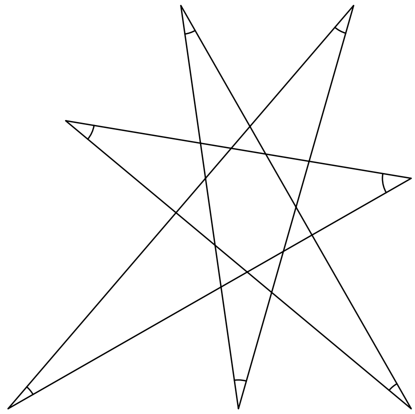
Problems for Contest 3

1. Suppose you have 9 evenly spaced dots in a circle on a piece of paper. You want to draw a 9-pointed star by connecting dots around the circle without lifting your pencil, skipping the same number of dots each time.

   Determine the number of different stars that can be drawn, if the regular nonagon does not count as a star.

2. Find (with proof) the units digit of the product of any 5 consecutive positive integers (*consecutive* means all in a row, like 5, 6, 7, 8, 9).

3. Consider the diagram on the right. What is the sum of the seven marked angles?



4. The 2020 members of the society of game theorists are holding the annual election for their leadership board. All members are initially on the board, and are ranked based on their qualifications. They start off by voting on whether to keep the board the same size; if they fail to get a strict majority, the member with the lowest rank is expelled. This process continues until they finally vote to keep the size of the board intact.

   It is common knowledge that, as each board member seeks to maximize their own influence, they seek to remain on the board while retaining as few other members as possible.

   At the end of this process, how many society members will remain on the leadership board?

5. Let $P(x)$ be a polynomial with positive real coefficients. Prove that

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

   holds for all positive real numbers x given that it holds for $x = 1$.

6. Let P be a polynomial with integer coefficients such that $P(2020) = P(2021) = 2021$. Prove that P has no integer roots.

7. Let
   - P be a point inside a triangle $\triangle ABC$,
   - $\triangle DEF$ be the pedal triangle of P, i.e., let D, E, F be the feet of the altitudes from P to BC, CA, AB, respectively,
   - I be the incenter of $\triangle ABC$, and
   - $\triangle XYZ$ be the Cevian triangle of I, i.e., X, Y, Z be the intersections of AI, BI, CI with BC, CA, AB, respectively.

   Show that there is a triangle with side lengths PD, PE, and PF if and only if P is inside $\triangle XYZ$.