Berkeley Math Circle: Monthly Contest 2

Due October 28, 2020

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 2 is due on October 28, 2020.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 2, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 2

1. The five-digit number $9A65B$ is divisible by 36, where A and B are digits. Find all possible values of A and B.

2. If x is a positive real number, find the smallest possible value of $2x + \frac{18}{x}$.

3. There are infinitely many bowls arranged on the number line, one at each integer. Initially each bowl has one fruit in it. In a move, one may take any fruit and move it to an adjacent bowl (bowls may hold more than one fruit, or no fruits at all).

   Is it possible that after 999 moves, every bowl still has exactly one fruit remaining?

4. In triangle ABC, $\angle A = 50°, \angle B = 60°$, and $\angle C = 70°$. A ray of light bounces from point D on BC to E on CA to F on AB and then back to D. Find the angles of $\triangle DEF$.

   (Light always takes the shortest path between points, meaning it bounces off an edge at equal angles.)

5. Aerith and Bob take turns picking a nonnegative integer, each time subtracting a (positive) divisor from the other's last number. The first person to pick 0 loses. For example, if Aerith reached 2020 on some turn, Bob could pick $2020 - 20 = 2000$, as 20 is a divisor of 2020.

   Continuing this example (with Aerith now picking a divisor of 2000), if both of them play optimally, who wins?

6. Find all perfect squares that can be written as the sum of two powers of 2.

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$(x - y)f(x + y) = xf(x) - yf(y).$$