Berkeley Math Circle: Monthly Contest 1

Due September 30, 2020

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 1 is due on September 30, 2020.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 1, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 1

1. A scalene triangle has side lengths which are all prime numbers. What is the smallest possible perimeter it could have?

2. In the addition equation

$$\begin{array}{cccc} & B & M & C \\ + & & M & C \\ \hline X & Y & 7 & 0 \end{array}$$

   each letter stands for a distinct digit and leading digits are not 0. Determine which digit each letter stands for.

3. Aerith has written down two letters on a piece of paper. Bob will pick a positive integer and she'll try to factor Bob's positive integer into two others, such that when spelled in English, one contains her first letter and the other her second.

   For example, if Aerith had chosen letters "v" and "w" and Bob chose "ten", she could spell it out as "fi$\underline{\text{v}}$e" times "t$\underline{\text{w}}$o," but would fail if Bob chose "twenty".

Show that there is a unique pair of letters Aerith can choose that allows her to succeed no matter what number Bob picks.

4. Let F be a point, d be a line, and let $\mathcal{P}$ be the parabola with focus F and directrix d, i.e. the set of points which are equidistant from F and d. Show that there is a line ℓ such that for any point A on $\mathcal{P}$, the circle with diameter AF is tangent to ℓ.

5. Let a, b, c be positive real numbers with $a + b + c = 1$. Prove that

$$a^4 + b^4 + c^4 \geq abc.$$

6. Aerith writes 50 consecutive positive integers in a circle on a whiteboard. Each minute after, she simultaneously replaces each number x with $2020a - x + 2020b$, where a and b were the numbers next to x. Can she choose her initial numbers such that she will never write down a negative number?

7. Given a fixed triangle $\triangle ABC$ and a point P, find the maximum value of

$$\frac{AB^2 + BC^2 + CA^2}{PA^2 + PB^2 + PC^2}.$$