Berkeley Math Circle: Monthly Contest 8

Due May 8, 2019

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 8 is due on May 8, 2019.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 8, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 8

1. Find an example of a triangle ABC with integer side lengths such that if M is the midpoint of $\overline{BC}$ and D is the foot of the altitude from A to $\overline{BC}$, then AD and AM have integer lengths too.

2. Find all pairs (m, n) of integers which satisfy the equation $m^2 + m = n^2 - 2n$.

3. Show that the numbers from 1 to 99 can be partitioned into two sets A and B with equal sum, and which satisfy $|A| = |B| + 1$ (i.e. A has one more element than B).

4. Let $s(n)$ denote the sum of the digits of a positive integer n. For example, $s(2019) = 2 + 0 + 1 + 9 = 12$. Find the 4-digit number n such which minimizes $\frac{n}{s(n)}$. (Leading zeros are not permitted.)

5. Suppose 4951 distinct points in the plane are given such that no four points are collinear. Show that it is possible to select 100 of the points for which no three points are collinear.

6. For which integers $n \geq 3$ does there exist a convex equiangular n-gon with rational side lengths which is not regular?

7. Three distinct circles Ω_1, Ω_2, Ω_3 cut three common chords concurrent at X. Consider two distinct circles Γ_1, Γ_2 which are internally tangent to all Ω_i. Prove that X lies on the line joining the centers of Γ_1 and Γ_2.