Berkeley Math Circle: Monthly Contest 7

Due April 10, 2019

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 7 is due on April 10, 2019.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 7, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 7

1. Several weights are given, each of which is not heavier than 1 lb. It is known that they cannot be divided into two groups such that the weight of each group is greater than 1 lb. Find the maximum possible total weight of these weights.

2. Find the value of the infinite continued fraction
$$1+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{\cdots}}}}.$$

3. Let C be a circle with center at the origin O of a system of rectangular coordinates, and let MON be the quarter circle of C in the first quadrant. Let PQ be an arc of C of fixed length that lies in the arc MN. Let K and L be the feet of the perpendiculars from P and Q to ON, and let V and W be the feet of the perpendiculars from P and Q to OM, respectively. Let A be the area of trapezoid $PKLQ$ and B the area of trapezoid $PVWQ$. Prove that $A+B$ does not depend on where arc PQ is chosen.

4. Prove that each nonnegative integer can be represented in the form $a^2 + b^2 - c^2$, where a, b, c are positive integers with $a < b < c$.

5. Let p and q be positive real numbers with $p + q < 1$. Teams A and B play a series of games. For each game, A wins with probability p, B wins with probability q, and they tie with probability $1 - p - q$. The series ends when one team has won two more games than the other, that team being declared the winner of the series. What is the probability that A wins the series?

6. If triangle ABC has perimeter 2, prove that not all its altitudes can exceed $1/\sqrt{3}$ in length.

7. Let $\sigma(n)$ denote the sum of the positive divisors of n. We say n is perfect if $\sigma(n) = 2n$. If n is a positive integer such that

$$\frac{\sigma(n)}{n} = \frac{5}{3},$$

show that $5n$ is an odd perfect number.