Berkeley Math Circle: Monthly Contest 6

Due March 13, 2019

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 6 is due on March 13, 2019.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 6, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 6

1. Three friends wish to divide five different tasks among themselves, such that every friend must handle at least one task. In how many different ways can this be done?

2. Let T be a triangle with area 1. We let T_1 be the medial triangle of T, i.e. the triangle whose vertices are the midpoints of sides of T. We then let T_2 be the medial triangle of T_1, T_3 the medial triangle of T_2, and so on. What is the sum of the areas of T_1, T_2, T_3, T_4, $\ldots$?

3. Prove that if x, y, z are positive real numbers, then

$$x^2 + 2y^2 + 3z^2 > xy + 3yz + zx.$$

4. Can the sum of three fourth powers end with the four digits 2019? (A fourth power is an integer of the form n^4, where n is an integer.)

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any real numbers x and y,

$$f(x + y) = \max(f(x), y) + \min(f(y), x).$$

6. We are given a family $\mathcal{F}$ of functions from the set $\{1, \dots, n\}$ to itself. A sequence $(f_1, \dots, f_k)$ of functions in $\mathcal{F}$ is said to be *good* if $f_k \circ f_{k-1} \circ \dots \circ f_1$ is a constant function. Prove that if there exists a good sequence, there exists one with $k \le n^3$.

7. Let ABC be an acute triangle with circumcircle γ and incenter I. Let D be the midpoint of minor arc $\widehat{BC}$ of γ. Let P be the reflection of the incenter of ABC over side BC. Suppose line DP meets γ again at a point Q on minor arc $\widehat{AB}$. Show that $AI = IQ$.