Berkeley Math Circle: Monthly Contest 5

Due February 13, 2019

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 5 is due on February 13, 2019.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 5, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 5

1. An artist paints identical dragons on two circular discs of the same size. On the first disc, the dragon covers the center, but on the second it is not. Show that it is possible to cut the second disc into two pieces that can be reassembled so the dragon covers the center.

2. Let n be a positive integer. Show that $2n+1$ and $4n^2+1$ are *relatively prime*, that is, their only common factor is 1.

3. Let P be a polynomial with positive real coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

   holds for $x = 1$, then it holds for every $x > 0$.

4. Let $\overline{ST}$ be a chord of a circle ω which is not a diameter, and let A be a fixed point on $\overline{ST}$. For which point X on minor arc $\widehat{ST}$ is the length AX minimized?

5. Show that $x^2 + y^2 = z^5 + z$ has infinitely many relatively prime integer solutions.

6. Let $ABCD$ be a cyclic convex quadrilateral such that $AD + BC = AB$. Prove that the bisectors of the angles ADC and BCD meet on the line AB.

7. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc},$$

and that equality occurs if and only if $a = b = c = 1$.