Berkeley Math Circle: Monthly Contest 4

Due January 23, 2019

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 4 is due on January 23, 2019.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 4, Problem 2
  Evan o'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 4

1. A certain lecture has finitely many students and at least two students. Every student fell asleep exactly once and woke up exactly once. Suppose that for any two students, there was some time at which both were asleep. Prove that there was a time at which all the students were asleep.

2. On a certain block, there are five houses in a line, which are to be painted red or green. If no two houses next to each other can be red, how many ways can you paint the houses?

3. Let ABC be a triangle with incenter I. Show that the circumcenter of $\triangle BIC$ lies on the circumcircle of $\triangle ABC$.

4. Let x, y, z be three real numbers. Prove the inequality

$$|x| + |y| + |z| - |x + y| - |y + z| - |z + x| + |x + y + z| \geq 0.$$

5. A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white and white becomes black). Determine the minimum number of mouse-clicks needed to make the chessboard all one color.

6. Let $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ be nonnegative real numbers such that $x_i + y_i = 1$ for each $i = 1, 2, \dots, n$. Prove that

$$(1 - x_1 x_2 \dots x_n)^m + (1 - y_1^m)(1 - y_2^m) \dots (1 - y_n^m) \geq 1,$$

where m is an arbitrary positive integer.

7. Let $P(x)$ be a polynomial with real coefficients so that $P(x) \geq 0$ for all real x. Prove that there exist polynomials $Q_1(x)$ and $Q_2(x)$ with real coefficients such that $P(x) = Q_1^2(x) + Q_2^2(x)$ for all x.