Berkeley Math Circle: Monthly Contest 3

Due December 4, 2019

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 3 is due on December 4, 2019.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 3, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

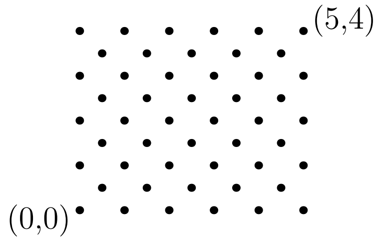
  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 3

1. Determine whether there exist three positive integers a, b, c such that $a+b$, $b+c$, and $c+a$ are all pairwise distinct prime numbers.

2. Given integers $m \geq n \geq 1$, we define $F_{m,n}$ as the set of all points (x, y) such that $0 \leq x \leq m$, $0 \leq y \leq n$, and $2x$, $2y$, and $x+y$ are all integers. For example, $F_{5,4}$ consists of 50 points and resembles the arrangement of stars on the American flag:



   (a) Find the number of points in $F_{m,n}$ in terms of m and n.

(b) Find all pairs (m, n) such that $F_{m,n}$ has exactly 5000 points.

3. Let $APBCD$ be a convex pentagon for which $ABCD$ is a square. Diagonals PD and AB meet at Q, while diagonals PC and AB meet at R. Prove that the sum of the areas of triangles PAQ and PBR equals the area of triangle DQR.

4. If you label your thumbs with the number 1, index fingers with the number 2, and so on up to 5 on your little fingers, then when you put your hands together with each finger touching the corresponding finger on the you earn a score of

$$1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 + 5 \cdot 5 = 55$$

which is the highest score you can get. If you turn your hands so that one thumb is on the other index finger, and so on, you'd have $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + 5 \cdot 1 = 45$.

(a) By turning your hands in this way, what is the smallest score you can get?

(b) If aliens with 12 fingers on each hand play this game, what is their highest and lowest possible score across the 12 possible turns?

(c) If aliens with n fingers on each hand play this game, what is their highest and lowest possible score across the n possible turns?

5. Suppose f is a function such that $f(xy + 1) = xf(y) - f(x) + 6$ for all real numbers x and y. Find all possible functions f that satisfy this equation and prove that no other functional solutions exist.

6. Let ABC be a nondegenerate triangle. Let A_1, B_1, C_1 be any points on lines BC, CA, AB, respectively. Let A_2, B_2, C_2 denote the midpoinst of AA_1, BB_1, CC_1, respectively.

   Prove that the points A_2, B_2 and C_2 are collinear if and only if one or more of A_1, B_1 and C_1 coincides with a vertex of the triangle ABC.

7. Show that there are infinitely many pairs of integers (x, y) satisfying

$$x^2 + y^2 + 2017 = 2019xy.$$