Berkeley Math Circle: Monthly Contest 2

Due October 30, 2019

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 2 is due on October 30, 2019.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 2, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

Problems for Contest 2

1. Do there exist five points in the plane, not all collinear, such that the distance between any pair is one of $\{1, 2, 3, \ldots 9\}$?

2. Consider three hockey pucks lying on a level sheet of ice; the pucks are not collinear. In a move, one may select any of the pucks and hit it so that it passes through the midpoint of the other two pucks. Determine whether it is possible, after 2019 such moves, for all pucks to be in their original positions.

3. We use the digits $1, 2, \ldots, 9$ once each to form two integers (e.g., 7419 and 82635). What two integers formed in this way have the greatest product? Prove your answer.

4. Let $ABCD$ be an isosceles trapezoid, and let E be the foot of the altitude from A to line BC. Prove that line DE passes through the centroid of $\triangle ABC$.

5. Let n, k, r be positive integers. Suppose we have a collection of sets $S_1, S_2, \ldots, S_r$, where each S_i is a subset of $\{1, 2, \ldots, n\}$ consisting of one or more consecutive integers. We say that such a collection is a *k-fold perfect cover* of $\{1, 2, \ldots, n\}$ if each element of $\{1, 2, \ldots, n\}$ occurs in exactly k of the sets S_i. (As an example, $S_1 = \{1\}$, $S_2 = \{1, 2, 3\}$, $S_3 = \{2\}$, $S_4 = \{3, 4, 5\}$, $S_5 = \{4, 5\}$ is a 2-fold perfect cover when $n = 5$.)

   Given a collection of sets which is a k-fold perfect cover of $\{1, 2, \ldots, n\}$, show that we can partition the collection into k subcollections, each of which is a 1-fold perfect cover of $\{1, 2, \ldots, n\}$.

6. Prove that if the line joining the circumcenter O and the incenter I is parallel to side BC of an acute triangle, then $\cos B + \cos C = 1$.

7. To play the lottery game Sum Thing, you choose five distinct numbers from 1 to 50, then the lottery master chooses five distinct numbers from 1 to 50. If there exist a nonempty subset of your five numbers and a nonempty subset of the lottery master's five numbers such that both subsets have the same sum, then you win.

   Can you choose five numbers that guarantee a win? Either demonstrate such a set, with a proof of validity, or prove that no such set exists.