# Berkeley Math Circle: Monthly Contest 1

Due October 2, 2019

## Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 1 is due on October 2, 2019.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 1, Problem 2
  Evan O'Dorney
  Grade 3, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules.

Enjoy solving these problems and good luck!

## Problems for Contest 1

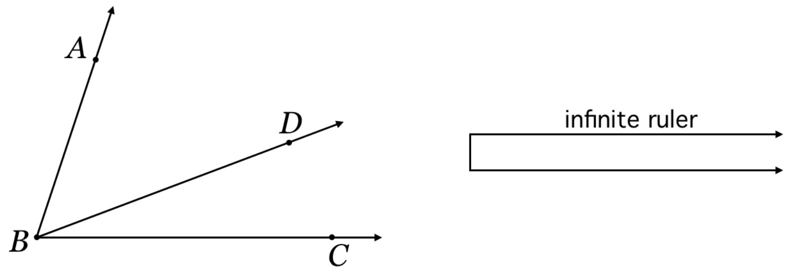
1. Which is larger, $A = 20^{19^{20}}$ or $B = 19^{20^{19}}$? (Here, $a^{b^c}$ means $a^{(b^c)}$ and not $(a^b)^c$.)

2. We wish to distribute 12 indistinguishable stones among 4 distinguishable boxes $B_1$, $B_2$, $B_3$, $B_4$. (It is permitted some boxes are empty.)
   (a) Over all ways to distribute the stones, what fraction of them have the property that the number of stones in every box is even?
   (b) Over all ways to distribute the stones, what fraction of them have the property that the number of stones in every box is odd?

3. Find the number of ordered pairs $(a, b)$ of positive integers such that $a$ and $b$ both divide $20^{19}$, but $ab$ does not.

4. An acute angle $\angle ABC$ and interior ray $BD$ are given, as shown. Laura is given an *infinite ruler* which consists of two parallel rays joined on one end by a segment perpendicular to both of them. One may place the infinite ruler onto the diagram

so that one of the infinite edges (marked with an arrow) passes through any two selected points in the diagram, or so that any edge of the ruler coincides with (i.e. exactly overlaps with) a portion of a segment, ray or line already in the diagram. Once the ruler is placed, one may draw any edge of the ruler "onto the diagram", in the usual fashion. One may also plot points where any two straight objects intersect.




Using only an infinite ruler, describe how to construct points $P$ on ray $BA$ and $Q$ on ray $BC$ such that ray $BD$ intersects ray $PQ$ at a point $R$ with $PR = 2(QR)$. Then prove that your construction works.

5. Let $a$, $b$, $c$ be positive real numbers. Assume that

$$\frac{a^{19}}{b^{19}} + \frac{b^{19}}{c^{19}} + \frac{c^{19}}{a^{19}} \leq \frac{a^{19}}{c^{19}} + \frac{b^{19}}{a^{19}} + \frac{c^{19}}{b^{19}}.$$

Prove that

$$\frac{a^{20}}{b^{20}} + \frac{b^{20}}{c^{20}} + \frac{c^{20}}{a^{20}} \leq \frac{a^{20}}{c^{20}} + \frac{b^{20}}{a^{20}} + \frac{c^{20}}{b^{20}}.$$

6. Let $ABC$ be a triangle with circumcircle $\Gamma$, whose incircle touches $BC$, $CA$, $AB$ at $D$, $E$, $F$. We draw a circle tangent to segment $BC$ at $D$ and to minor arc $\widehat{BC}$ of $\Gamma$ at the point $A_1$. Define $B_1$ and $C_1$ in a similar way. Prove that lines $A_1D$, $B_1E$, $C_1F$ are concurrent.

7. Are there positive integers $a$ and $b$ satisfying $a^2 - 23 = b^{11}$?